CHAPTER 5

Trigonometric Functions and Graphs

You have seen different types of functions and how these functions can mathematically model the real world. Many sinusoidal and periodic patterns occur within nature. Movement on the surface of Earth, such as earthquakes, and stresses within Earth can cause rocks to fold into a sinusoidal pattern. Geologists and structural engineers study models of trigonometric functions to help them understand these formations. In this chapter, you will study trigonometric functions for which the function values repeat at regular intervals.

Key Terms

periodic function
period
sinusoidal curve
amplitude
vertical displacement
phase shift

Career Link

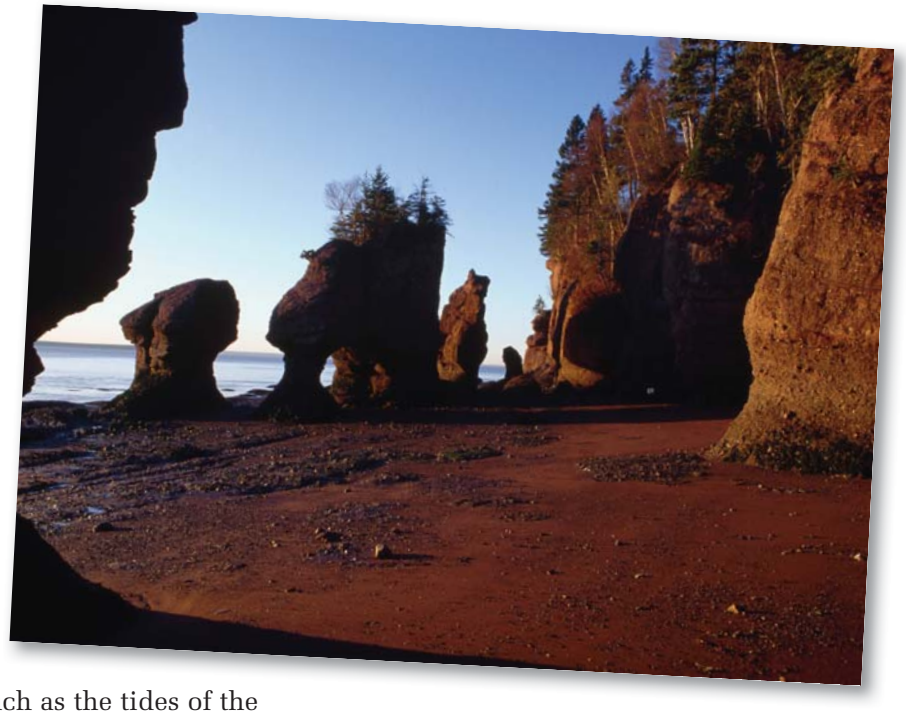
A geologist studies the composition, structure, and history of Earth's surface to determine the processes affecting the development of Earth. Geologists apply their knowledge of physics, chemistry, biology, and mathematics to explain these phenomena. Geological engineers apply geological knowledge to projects such as dam, tunnel, and building construction.

Web Link

To learn more about a career as a geologist, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

5.1

Graphing Sine and Cosine Functions



The Hopewell Rocks on the Bay of Fundy coastline are sculpted by the cyclic tides.

Focus on...

- sketching the graphs of $y = \sin x$ and $y = \cos x$
- determining the characteristics of the graphs of $y = \sin x$ and $y = \cos x$
- demonstrating an understanding of the effects of vertical and horizontal stretches on the graphs of sinusoidal functions
- solving a problem by analysing the graph of a trigonometric function

Many natural phenomena are cyclic, such as the tides of the ocean, the orbit of Earth around the Sun, and the growth and decline in animal populations. What other examples of cyclic natural phenomena can you describe?

You can model these types of natural behaviour with periodic functions such as sine and cosine functions.

Did You Know?

The Bay of Fundy, between New Brunswick and Nova Scotia, has the highest tides in the world. The highest recorded tidal range is 17 m at Burntcoat Head, Nova Scotia.

Investigate the Sine and Cosine Functions

Materials

- grid paper
- ruler

1. a) Copy and complete the table. Use your knowledge of special angles to determine exact values for each trigonometric ratio. Then, determine the approximate values, to two decimal places. One row has been completed for you.

Angle, θ	$y = \sin \theta$	$y = \cos \theta$
0		
$\frac{\pi}{6}$	$\frac{1}{2} = 0.50$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{\pi}{4}$		
$\frac{\pi}{3}$		
$\frac{\pi}{2}$		

b) Extend the table to include multiples of the special angles in the other three quadrants.

2. a) Graph $y = \sin\theta$ on the interval $\theta \in [0, 2\pi]$

b) Summarize the following characteristics of the function $y = \sin\theta$.

- the maximum value and the minimum value
- the interval over which the pattern of the function repeats
- the zeros of the function in the interval $\theta \in [0, 2\pi]$
- the y-intercept
- the domain and range

3. Graph $y = \cos\theta$ on the interval $\theta \in [0, 2\pi]$ and create a summary similar to the one you developed in step 2b).

Reflect and Respond

4. a) Suppose that you extended the graph of $y = \sin\theta$ to the right of 2π. Predict the shape of the graph. Use a calculator to investigate a few points to the right of 2π. At what value of θ will the next cycle end?

b) Suppose that you extended the graph of $y = \sin\theta$ to the left of 0. Predict the shape of the graph. Use a calculator to investigate a few points to the left of 0. At what value of θ will the next cycle end?

5. Repeat step 4 for $y = \cos\theta$.

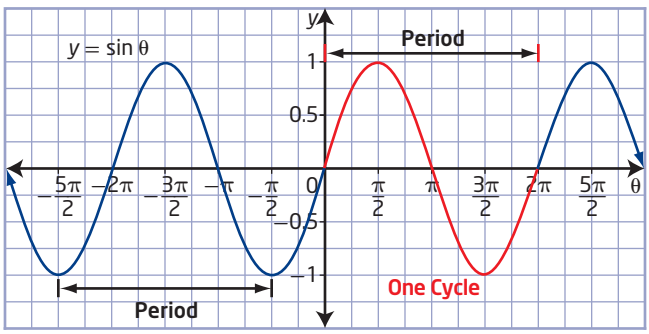
Did You Know?

The sine function is based upon one of the trigonometric ratios originally calculated by the astronomer Hipparchus of Nicaea in the second century B.C.E. He was trying to make sense of the movement of the stars and the moon in the night sky.

Link the Ideas

Sine and cosine functions are **periodic functions**. The values of these functions repeat over a specified **period**.

A sine graph is a graph of the function $y = \sin\theta$. You can also describe a sine graph as a **sinusoidal curve**.



Trigonometric functions are sometimes called circular because they are based on the unit circle.

periodic function

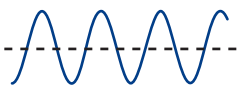
- a function that repeats itself over regular intervals (cycles) of its domain

period

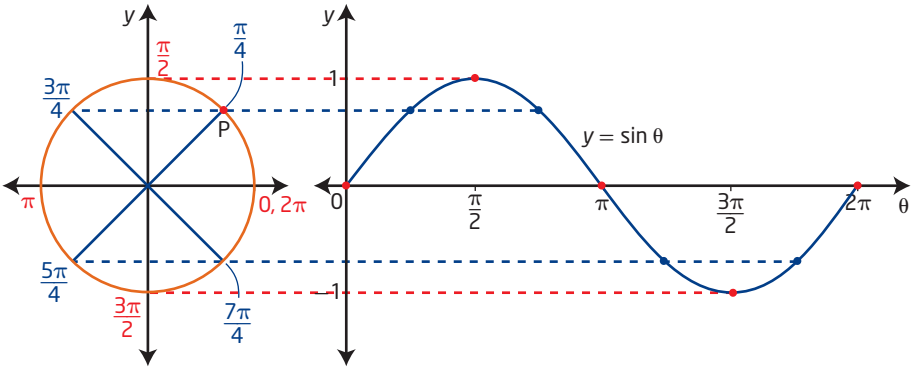
- the length of the interval of the domain over which a graph repeats itself
- the horizontal length of one cycle on a periodic graph

sinusoidal curve

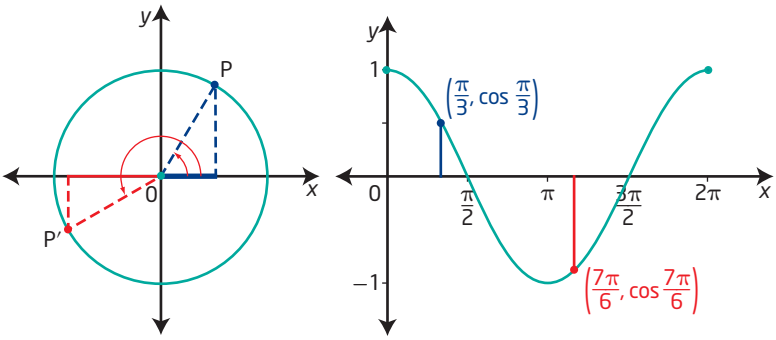
- the name given to a curve that fluctuates back and forth like a sine graph
- a curve that oscillates repeatedly up and down from a centre line



The sine function, $y = \sin \theta$, relates the measure of angle θ in standard position to the y-coordinate of the point P where the terminal arm of the angle intersects the unit circle.



The cosine function, $y = \cos \theta$, relates the measure of angle θ in standard position to the x-coordinate of the point P where the terminal arm of the angle intersects the unit circle.



The coordinates of point P repeat after point P travels completely around the unit circle. The unit circle has a circumference of 2π. Therefore, the smallest distance before the cycle of values for the functions $y = \sin \theta$ or $y = \cos \theta$ begins to repeat is 2π. This distance is the period of $\sin \theta$ and $\cos \theta$.

Example 1

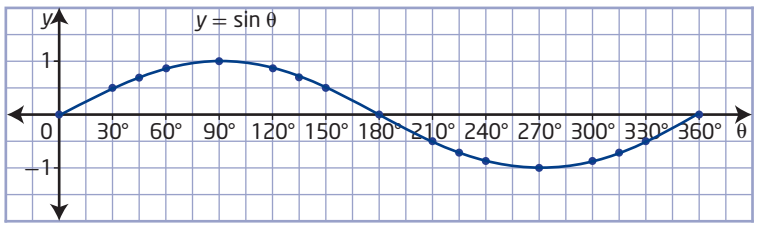
Graph a Periodic Function

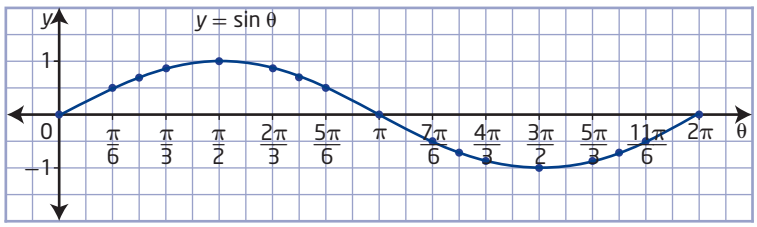
Sketch the graph of $y = \sin \theta$ for $0° \leq \theta \leq 360°$ or $0 \leq \theta \leq 2\pi$. Describe its characteristics.

Solution

To sketch the graph of the sine function for $0° \leq \theta \leq 360°$ or $0 \leq \theta \leq 2\pi$, select values of θ and determine the corresponding values of $\sin \theta$. Plot the points and join them with a smooth curve.

θ	Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
	Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin θ		0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0





From the graph of the sine function, you can make general observations about the characteristics of the sine curve:

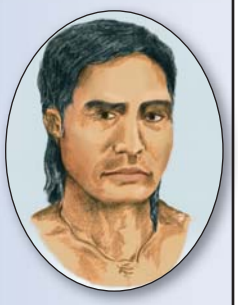
- The curve is periodic.
- The curve is continuous.
- The domain is $\{\theta \mid \theta \in R\}$.
- The range is $\{y \mid -1 \leq y \leq 1, y \in R\}$.
- The maximum value is $+1$.
- The minimum value is -1.
- The **amplitude** of the curve is 1.
- The period is 360° or 2π.
- The y-intercept is 0.
- In degrees, the θ-intercepts are
  …, −540°, −360°, −180°, 0°, 180°, 360°, …, or $180°n$,
  where $n \in I$.
  The θ-intercepts, in radians, are
  …, -3π, -2π, $-\pi$, 0, π, 2π, …, or $n\pi$,
  where $n \in I$.

Which points would you determine to be the key points for sketching a graph of the sine function?

Look for a pattern in the values.

Did You Know?

The Indo-Asian mathematician Aryabhata (476–550) made tables of half-chords that are now known as sine and cosine tables.



amplitude (of a sinusoidal function)

- the maximum vertical distance the graph of a sinusoidal function varies above and below the horizontal central axis of the curve

Your Turn

Sketch the graph of $y = \cos\theta$ for $0° \leq \theta \leq 360°$. Describe its characteristics.

Example 2

Determine the Amplitude of a Sine Function

Any function of the form $y = af(x)$ is related to $y = f(x)$ by a vertical stretch of a factor $|a|$ about the x-axis, including the sine and cosine functions. If $a < 0$, the function is also reflected in the x-axis.

a) On the same set of axes, graph $y = 3 \sin x$, $y = 0.5 \sin x$, and $y = -2 \sin x$ for $0 \leq x \leq 2\pi$.

b) State the amplitude for each function.

c) Compare each graph to the graph of $y = \sin x$. Consider the period, amplitude, domain, and range.

Solution

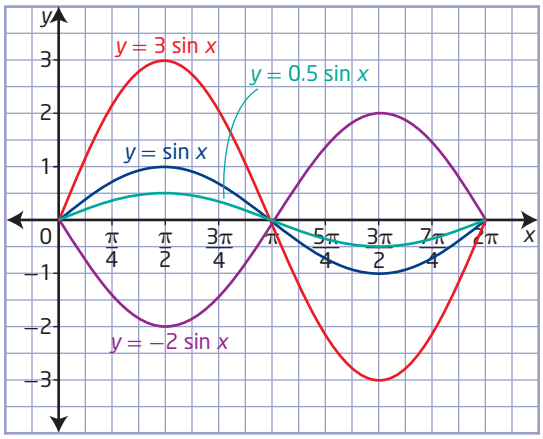
a) Method 1: Graph Using Transformations

Sketch the graph of $y = \sin x$.

For the graph of $y = 3 \sin x$, apply a vertical stretch by a factor of 3.

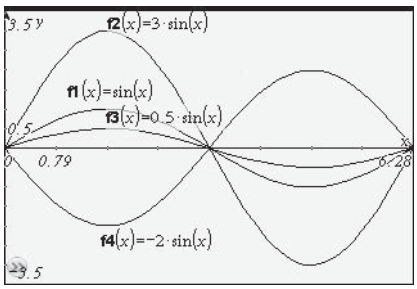
For the graph of $y = 0.5 \sin x$, apply a vertical stretch by a factor of 0.5.

For the graph of $y = -2 \sin x$, reflect in the x-axis and apply a vertical stretch by a factor of 2.



Method 2: Use a Graphing Calculator

Select radian mode.



Use the following window settings:
x: $\left[0, 2\pi, \frac{\pi}{4}\right]$
y: $[-3.5, 3.5, 0.5]$

b) Determine the amplitude of a sine function using the formula

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}.$$

The amplitude of $y = \sin x$ is $\frac{1 - (-1)}{2}$, or 1.

The amplitude of $y = 3 \sin x$ is $\frac{3 - (-3)}{2}$, or 3.

The amplitude of $y = 0.5 \sin x$ is $\frac{0.5 - (-0.5)}{2}$, or 0.5.

The amplitude of $y = -2 \sin x$ is $\frac{2 - (-2)}{2}$, or 2.

How is the amplitude related to the range of the function?

c)

Function	Period	Amplitude	Specified Domain	Range
$y = \sin x$	2π	1	$\{x \mid 0 \le x \le 2\pi, x \in \mathbb{R}\}$	$\{y \mid -1 \le y \le 1, y \in \mathbb{R}\}$
$y = 3 \sin x$	2π	3	$\{x \mid 0 \le x \le 2\pi, x \in \mathbb{R}\}$	$\{y \mid -3 \le y \le 3, y \in \mathbb{R}\}$
$y = 0.5 \sin x$	2π	0.5	$\{x \mid 0 \le x \le 2\pi, x \in \mathbb{R}\}$	$\{y \mid -0.5 \le y \le 0.5, y \in \mathbb{R}\}$
$y = -2 \sin x$	2π	2	$\{x \mid 0 \le x \le 2\pi, x \in \mathbb{R}\}$	$\{y \mid -2 \le y \le 2, y \in \mathbb{R}\}$

Changing the value of a affects the amplitude of a sinusoidal function. For the function $y = a \sin x$, the amplitude is $|a|$.

Your Turn

a) On the same set of axes, graph $y = 6 \cos x$ and $y = -4 \cos x$ for $0 \le x \le 2\pi$.

b) State the amplitude for each graph.

c) Compare your graphs to the graph of $y = \cos x$. Consider the period, amplitude, domain, and range.

d) What is the amplitude of the function $y = 1.5 \cos x$?

Period of $y = \sin bx$ or $y = \cos bx$

The graph of a function of the form $y = \sin bx$ or $y = \cos bx$ for $b \neq 0$ has a period different from 2π when $|b| \neq 1$. To show this, remember that $\sin bx$ or $\cos bx$ will take on all possible values as bx ranges from 0 to 2π. Therefore, to determine the period of either of these functions, solve the compound inequality as follows.

$0 \le x \le 2\pi$ Begin with the interval of one cycle of $y = \sin x$ or $y = \cos x$.

$0 \le |b|x \le 2\pi$ Replace x with $|b|x$ for the interval of one cycle of $y = \sin bx$ or $y = \cos bx$.

$0 \le x \le \frac{2\pi}{|b|}$ Divide by $|b|$.

Solving this inequality determines the length of a cycle for the sinusoidal curve, where the start of a cycle of $y = \sin bx$ is 0 and the end is $\frac{2\pi}{|b|}$. Determine the period, or length of the cycle, by finding the distance from 0 to $\frac{2\pi}{|b|}$. Thus, the period for $y = \sin bx$ or $y = \cos bx$ is $\frac{2\pi}{|b|}$, in radians, or $\frac{360°}{|b|}$, in degrees.

Why do you use $|b|$ to determine the period?

Example 3

Determine the Period of a Sine Function

Any function of the form $y = f(bx)$ is related to $y = f(x)$ by a horizontal stretch by a factor of $\frac{1}{|b|}$ about the y-axis, including the sine and cosine functions. If $b < 0$, then the function is also reflected in the y-axis.

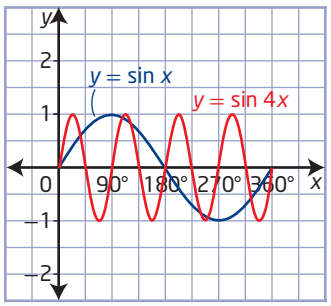
a) Sketch the graph of the function $y = \sin 4x$ for $0 \leq x \leq 360°$. State the period of the function and compare the graph to the graph of $y = \sin x$.

b) Sketch the graph of the function $y = \sin \frac{1}{2}x$ for $0 \leq x \leq 4\pi$. State the period of the function and compare the graph to the graph of $y = \sin x$.

Solution

a) Sketch the graph of $y = \sin x$.

For the graph of $y = \sin 4x$, apply a horizontal stretch by a factor of $\frac{1}{4}$.



To find the period of a function, start from any point on the graph (for example, the y-intercept) and determine the length of the interval until one cycle is complete.

From the graph of $y = \sin 4x$, the period is 90°.

You can also determine this using the formula Period $= \frac{360°}{|b|}$.

$$\text{Period} = \frac{360°}{|b|}$$

$$\text{Period} = \frac{360°}{|4|}$$ Substitute 4 for b.

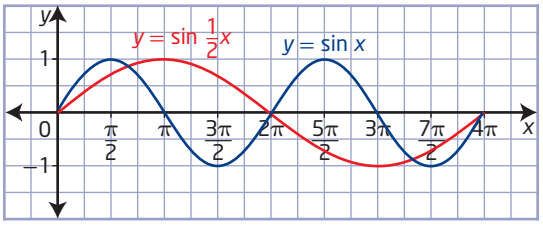
$$\text{Period} = \frac{360°}{4}$$

$$\text{Period} = 90°$$

Compared to the graph of $y = \sin x$, the graph of $y = \sin 4x$ has the same amplitude, domain, and range, but a different period.

b) Sketch the graph of $y = \sin x$.

For the graph of $y = \sin \frac{1}{2}x$, apply a horizontal stretch by a factor of 2.



From the graph, the period for $y = \sin \frac{1}{2}x$ is 4π.
Using the formula,

$$\text{Period} = \frac{2\pi}{|b|}$$

$$\text{Period} = \frac{2\pi}{\left|\frac{1}{2}\right|} \qquad \text{Substitute } \tfrac{1}{2} \text{ for } b.$$

$$\text{Period} = \frac{2\pi}{\frac{1}{2}}$$

$$\text{Period} = 4\pi$$

Compared to the graph of $y = \sin x$, the graph of $y = \sin \frac{1}{2}x$ has the same amplitude, domain, and range, but a different period.

Changing the value of b affects the period of a sinusoidal function.

Your Turn

a) Sketch the graph of the function $y = \cos 3x$ for $0 \le x \le 360°$. State the period of the function and compare the graph to the graph of $y = \cos x$.

b) Sketch the graph of the function $y = \cos \frac{1}{3}x$ for $0 \le x \le 6\pi$. State the period of the function and compare the graph to the graph of $y = \cos x$.

c) What is the period of the graph of $y = \cos(-3x)$?

Example 4

Sketch the Graph of $y = a \cos bx$

a) Sketch the graph of $y = -3 \cos 2x$ for at least one cycle.

b) Determine

- the amplitude
- the period
- the maximum and minimum values
- the x-intercepts and the y-intercept
- the domain and range

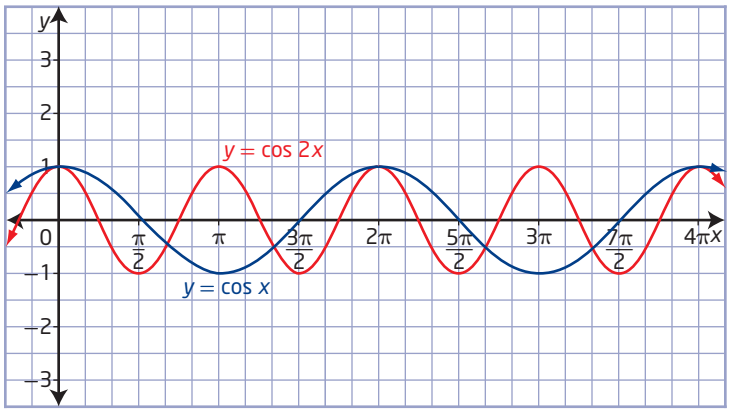
Solution

a) Method 1: Graph Using Transformations

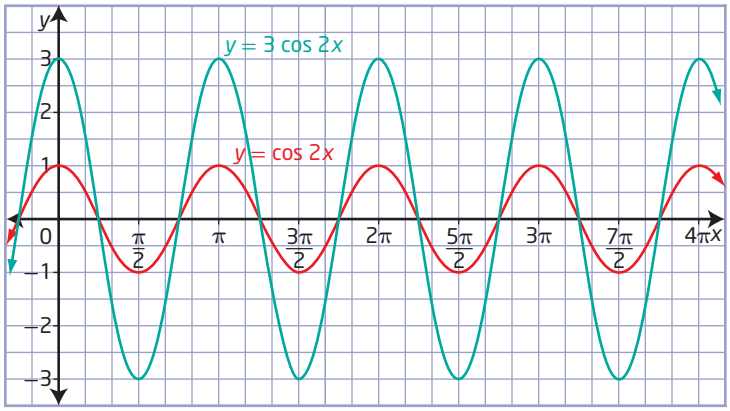
Compared to the graph of $y = \cos x$, the graph of $y = -3 \cos 2x$ is stretched horizontally by a factor of $\frac{1}{2}$ about the y-axis, stretched vertically by a factor of 3 about the x-axis, and reflected in the x-axis.

Begin with the graph of $y = \cos x$. Apply a horizontal stretch of $\frac{1}{2}$ about the y-axis.

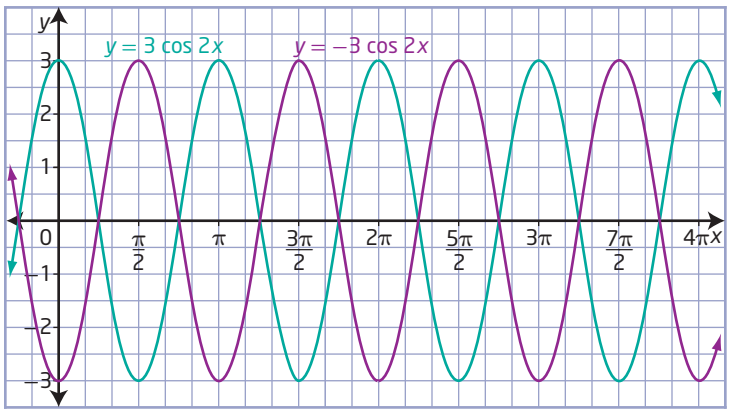
Why is the horizontal stretch by a factor of $\frac{1}{2}$?



Then, apply a vertical stretch by a factor of 3.



Finally, reflect the graph of $y = 3 \cos 2x$ in the x-axis.



Method 2: Graph Using Key Points

This method is based on the fact that one cycle of a cosine function $y = \cos bx$, from 0 to $\frac{2\pi}{|b|}$, includes two x-intercepts, two maximums, and a minimum. These five points divide the period into quarters.

Compare $y = -3 \cos 2x$ to $y = a \cos bx$.

Since $a = -3$, the amplitude is $|-3|$, or 3. Thus, the maximum value is 3 and the minimum value is -3.

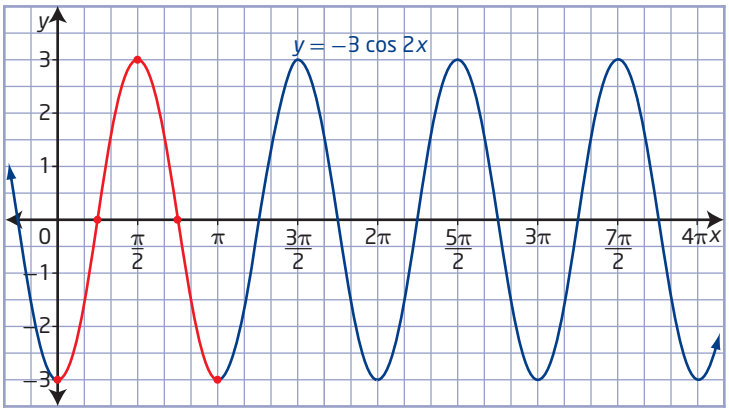
Since $b = 2$, the period is $\frac{2\pi}{|2|}$, or π. One cycle will start at $x = 0$ and end at $x = \pi$. Divide this cycle into four equal segments using the values 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, and π for x.

How do you know where the maximums or minimums will occur?

Why are there two minimums instead of two maximums?

The key points are $(0, -3)$, $\left(\frac{\pi}{4}, 0\right)$, $\left(\frac{\pi}{2}, 3\right)$, $\left(\frac{3\pi}{4}, 0\right)$, and $(\pi, -3)$.

Connect the points in a smooth curve and sketch the graph through one cycle. The graph of $y = -3 \cos 2x$ repeats every π units in either direction.



b) The amplitude of $y = -3 \cos 2x$ is 3.
The period is π.
The maximum value is 3.
The minimum value is -3
The y-intercept is -3.
The x-intercepts are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$ or $\frac{\pi}{4} + \frac{\pi}{2}n$, $n \in \mathrm{I}$.
The domain of the function is $\{x \mid x \in \mathrm{R}\}$.
The range of the function is $\{y \mid -3 \leq y \leq 3, y \in \mathrm{R}\}$.

Your Turn

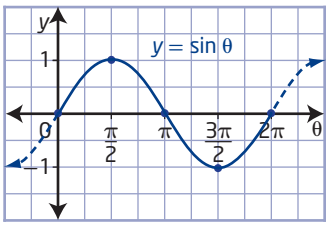
a) Graph $y = 3 \sin 4x$, showing at least two cycles.

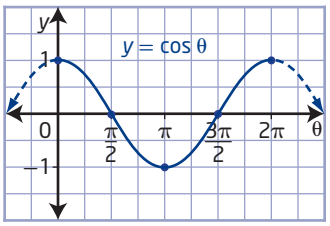
b) Determine

- the amplitude
- the period
- the maximum and minimum values
- the x-intercepts and the y-intercept
- the domain and range

Key Ideas

- To sketch the graphs of $y = \sin \theta$ and $y = \cos \theta$ for $0° \leq \theta \leq 360°$ or $0 \leq \theta \leq 2\pi$, determine the coordinates of the key points representing the θ-intercepts, maximum(s), and minimum(s).





How are the characteristics different for $y = \cos \theta$?

The maximum value is $+1$.
The minimum value is -1.
The amplitude is 1.
The period is 2π.
The y-intercept is 0.
The θ-intercepts for the cycle shown are 0, π, and 2π.
The domain of $y = \sin \theta$ is $\{\theta \mid \theta \in R\}$.
The range of $y = \sin \theta$ is $\{y \mid -1 \leq y \leq 1, y \in R\}$.

- Determine the amplitude and period of a sinusoidal function of the form $y = a \sin bx$ or $y = a \cos bx$ by inspecting graphs or directly from the sinusoidal function.
  - You can determine the amplitude using the formula
    $\text{Amplitude} = \dfrac{\text{maximum value} - \text{minimum value}}{2}$.
    The amplitude is given by $|a|$.
    You can change the amplitude of a function by varying the value of a.

How can you determine the amplitude from the graph of the sine function? cosine function?

  - The period is the horizontal length of one cycle on the graph of a function. It is given by $\dfrac{2\pi}{|b|}$ or $\dfrac{360°}{|b|}$.
    You can change the period of a function by varying the value of b.

How can you identify the period on the graph of a sine function? cosine function?

Check Your Understanding

Practise

1. **a)** State the five key points for $y = \sin x$ that occur in one complete cycle from 0 to 2π.

b) Use the key points to sketch the graph of $y = \sin x$ for $-2\pi \leq x \leq 2\pi$. Indicate the key points on your graph.

c) What are the x-intercepts of the graph?

d) What is the y-intercept of the graph?

e) What is the maximum value of the graph? the minimum value?

2. **a)** State the five key points for $y = \cos x$ that occur in one complete cycle from 0 to 2π.

b) Use the key points to sketch a graph of $y = \cos x$ for $-2\pi \leq x \leq 2\pi$. Indicate the key points on your graph.

c) What are the x-intercepts of the graph?

d) What is the y-intercept of the graph?

e) What is the maximum value of the graph? the minimum value?

3. Copy and complete the table of properties for $y = \sin x$ and $y = \cos x$ for all real numbers.

Property	$y = \sin x$	$y = \cos x$
maximum		
minimum		
amplitude		
period		
domain		
range		
y-intercept		
x-intercepts		

4. State the amplitude of each periodic function. Sketch the graph of each function.

a) $y = 2 \sin \theta$ **b)** $y = \frac{1}{2} \cos \theta$

c) $y = -\frac{1}{3} \sin x$ **d)** $y = -6 \cos x$

5. State the period for each periodic function, in degrees and in radians. Sketch the graph of each function.

a) $y = \sin 4\theta$ **b)** $y = \cos \frac{1}{3}\theta$

c) $y = \sin \frac{2}{3}x$ **d)** $y = \cos 6x$

Apply

6. Match each function with its graph.

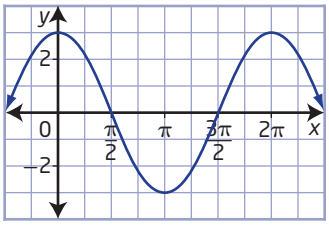
a) $y = 3 \cos x$

b) $y = \cos 3x$

c) $y = -\sin x$

d) $y = -\cos x$

A



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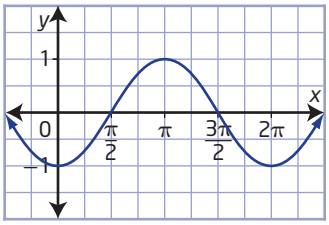

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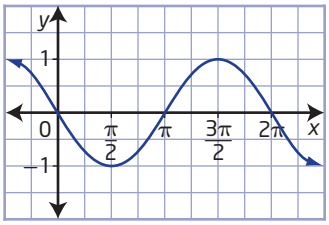
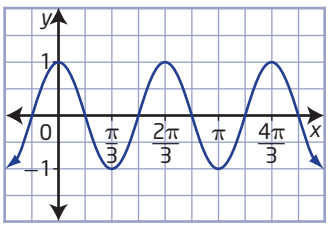


D



7. Determine the amplitude of each function. Then, use the language of transformations to describe how each graph is related to the graph of $y = \sin x$.

a) $y = 3 \sin x$ **b)** $y = -5 \sin x$

c) $y = 0.15 \sin x$ **d)** $y = -\frac{2}{3} \sin x$

8. Determine the period (in degrees) of each function. Then, use the language of transformations to describe how each graph is related to the graph of $y = \cos x$.

a) $y = \cos 2x$ **b)** $y = \cos(-3x)$

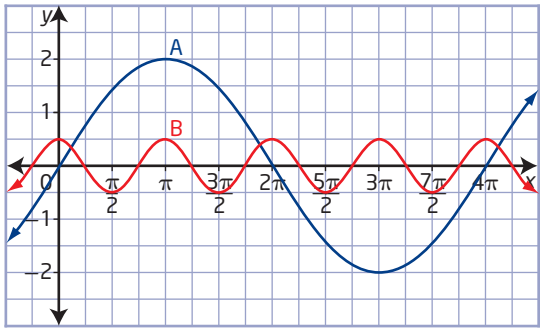
c) $y = \cos \frac{1}{4}x$ **d)** $y = \cos \frac{2}{3}x$

9. Without graphing, determine the amplitude and period of each function. State the period in degrees and in radians.

a) $y = 2 \sin x$ **b)** $y = -4 \cos 2x$

c) $y = \frac{5}{3} \sin\left(-\frac{2}{3}x\right)$ **d)** $y = 3 \cos \frac{1}{2}x$

10. a) Determine the period and the amplitude of each function in the graph.



b) Write an equation in the form $y = a \sin bx$ or $y = a \cos bx$ for each function.

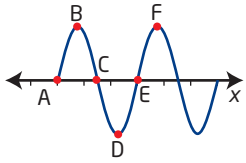
c) Explain your choice of either sine or cosine for each function.

11. Sketch the graph of each function over the interval $[-360°, 360°]$. For each function, clearly label the maximum and minimum values, the x-intercepts, the y-intercept, the period, and the range.

a) $y = 2 \cos x$ **b)** $y = -3 \sin x$

c) $y = \frac{1}{2} \sin x$ **d)** $y = -\frac{3}{4} \cos x$

12. The points indicated on the graph shown represent the x-intercepts and the maximum and minimum values.



a) Determine the coordinates of points B, C, D, and E if $y = 3 \sin 2x$ and A has coordinates $(0, 0)$.

b) Determine the coordinates of points C, D, E, and F if $y = 2 \cos x$ and B has coordinates $(0, 2)$.

c) Determine the coordinates of points B, C, D, and E if $y = \sin \frac{1}{2}x$ and A has coordinates $(-4\pi, 0)$.

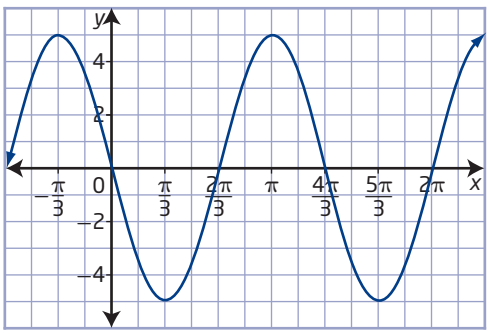
13. The second harmonic in sound is given by $f(x) = \sin 2x$, while the third harmonic is given by $f(x) = \sin 3x$. Sketch the curves and compare the graphs of the second and third harmonics for $-2\pi \leq x \leq 2\pi$.

Did You Know?

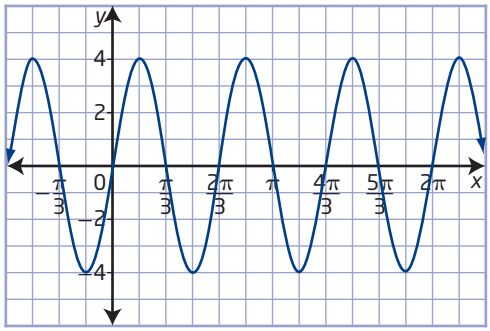
A harmonic is a wave whose frequency is an integral multiple of the fundamental frequency. The fundamental frequency of a periodic wave is the inverse of the period length.

14. Sounds heard by the human ear are vibrations created by different air pressures. Musical sounds are regular or periodic vibrations. Pure tones will produce single sine waves on an oscilloscope. Determine the amplitude and period of each single sine wave shown.

a)



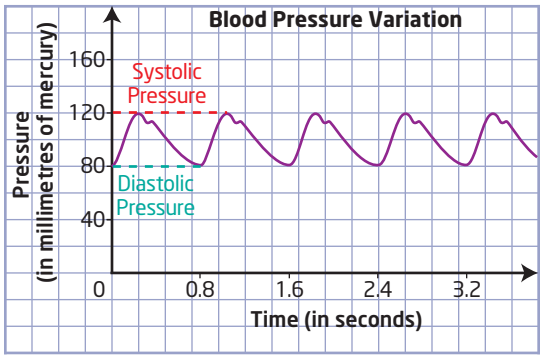
b)



Did You Know?

Pure tone audiometry is a hearing test used to measure the hearing threshold levels of a patient. This test determines if there is hearing loss. Pure tone audiometry relies on a patient's response to pure tone stimuli.

15. Systolic and diastolic pressures mark the upper and lower limits in the changes in blood pressure that produce a pulse. The length of time between the peaks relates to the period of the pulse.



a) Determine the period and amplitude of the graph.

b) Determine the pulse rate (number of beats per minute) for this person.

16. **MINI LAB** Follow these steps to draw a sine curve.

Materials

- paper
- protractor
- compass
- ruler
- grid paper

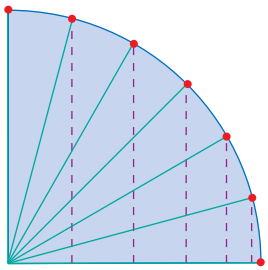
Step 1 Draw a large circle.

a) Mark the centre of the circle.

b) Use a protractor and mark every 15° from 0° to 180° along the circumference of the circle.

c) Draw a line radiating from the centre of the circle to each mark.

d) Draw a vertical line to complete a right triangle for each of the angles that you measured.



Step 2 Recall that the sine ratio is the length of the opposite side divided by the length of the hypotenuse. The hypotenuse of each triangle is the radius of the circle. Measure the length of the opposite side for each triangle and complete a table similar to the one shown.

Angle, x	Opposite	Hypotenuse	$\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$
0°			
15°			
30°			
45°			

Step 3 Draw a coordinate grid on a sheet of grid paper.

a) Label the x-axis from 0° to 360° in increments of 15°.

b) Label the y-axis from -1 to $+1$.

c) Create a scatter plot of points from your table. Join the dots with a smooth curve.

Step 4 Use one of the following methods to complete one cycle of the sine graph:

- complete the diagram from 180° to 360°
- extend the table by measuring the lengths of the sides of the triangle
- use the symmetry of the sine curve to complete the cycle

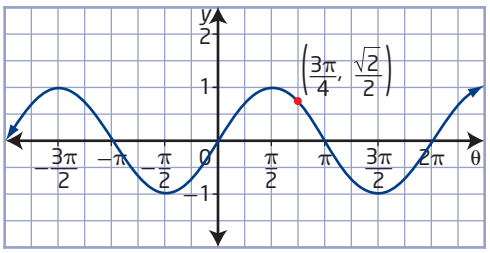
17. Sketch one cycle of a sinusoidal curve with the given amplitude and period and passing through the given point.

a) amplitude 2, period 180°, point (0, 0)

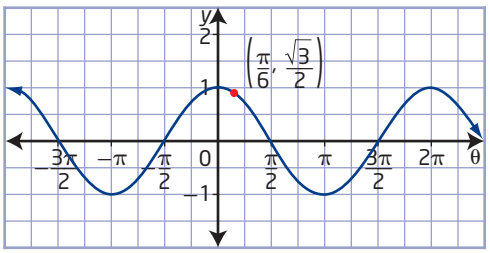
b) amplitude 1.5, period 540°, point (0, 0)

18. The graphs of $y = \sin\theta$ and $y = \cos\theta$ show the coordinates of one point. Determine the coordinates of four other points on the graph with the same y-coordinate as the point shown. Explain how you determined the θ-coordinates.

a)



b)



19. Graph $y = \sin\theta$ and $y = \cos\theta$ on the same set of axes for $-2\pi \le \theta \le 2\pi$.

a) How are the two graphs similar?

b) How are they different?

c) What transformation could you apply to make them the same graph?

Extend

20. If $y = f(x)$ has a period of 6, determine the period of $y = f\left(\frac{1}{2}x\right)$.

21. Determine the period, in radians, of each function using two different methods.

a) $y = -2\sin 3x$

b) $y = -\frac{2}{3}\cos\frac{\pi}{6}x$

22. If $\sin\theta = 0.3$, determine the value of $\sin\theta + \sin(\theta + 2\pi) + \sin(\theta + 4\pi)$.

23. Consider the function $y = \sqrt{\sin x}$.

a) Use the graph of $y = \sin x$ to sketch a prediction for the shape of the graph of $y = \sqrt{\sin x}$.

b) Use graphing technology or grid paper and a table of values to check your prediction. Resolve any differences.

c) How do you think the graph of $y = \sqrt{\sin x + 1}$ will differ from the graph of $y = \sqrt{\sin x}$?

d) Graph $y = \sqrt{\sin x + 1}$ and compare it to your prediction.

24. Is the function $f(x) = 5\cos x + 3\sin x$ sinusoidal? If it is sinusoidal, state the period of the function.

Did You Know?

In 1822, French mathematician Joseph Fourier discovered that any wave could be modelled as a combination of different types of sine waves. This model applies even to unusual waves such as square waves and highly irregular waves such as human speech. The discipline of reducing a complex wave to a combination of sine waves is called Fourier analysis and is fundamental to many of the sciences.

Create Connections

C1 **MINI LAB** Explore the relationship between the unit circle and the sine and cosine graphs with a graphing calculator.

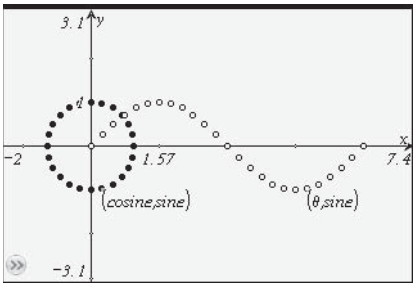
Step 1 In the first list, enter the angle values from 0 to 2π by increments of $\frac{\pi}{12}$. In the second and third lists, calculate the cosine and sine of the angles in the first list, respectively.

	A θ	B cosine	C sine	D	E
◆	=seq(x, x,0	=cos(θ)	=sin(θ)		
1	0.	1.	0.		
2	0.261799	0.965926	0.258819		
3	0.523599	0.866025	0.5		
4	0.785398	0.707107	0.707107		
5	1.0472	0.5	0.866025		
6	1.309	0.258819	0.965926		
	A1	=0.			

Step 2 Graph the second and third lists for the unit circle.

Step 3 Graph the first and third lists for the sine curve.

Step 4 Graph the first and second lists for the cosine curve.



Step 5 **a)** Use the trace feature on the graphing calculator and trace around the unit circle. What do you notice about the points that you trace? What do they represent?

b) Move the cursor to trace the sine or cosine curve. How do the points on the graph of the sine or cosine curve relate to the points on the unit circle? Explain.

C2 The value of $(\cos \theta)^2 + (\sin \theta)^2$ appears to be constant no matter the value of θ. What is the value of the constant? Why is the value constant? (Hint: Use the unit circle and the Pythagorean theorem in your explanation.)

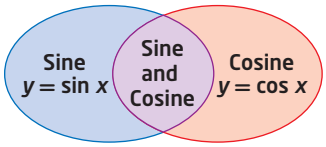
C3 The graph of $y = f(x)$ is sinusoidal with a period of 40° passing through the point (4, 0). Decide whether each of the following can be determined from this information, and justify your answer.

a) $f(0)$

b) $f(4)$

c) $f(84)$

C4 Identify the regions that each of the following characteristics fall into.



a) domain $\{x \mid x \in \mathrm{R}\}$

b) range $\{y \mid -1 \le y \le 1, y \in \mathrm{R}\}$

c) period is 2π

d) amplitude is 1

e) x-intercepts are $n(180°)$, $n \in \mathrm{I}$

f) x-intercepts are $90° + n(180°)$, $n \in \mathrm{I}$

g) y-intercept is 1

h) y-intercept is 0

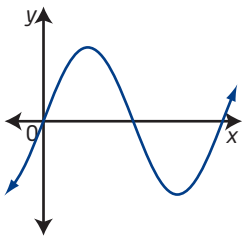
i) passes through point (0, 1)

j) passes through point (0, 0)

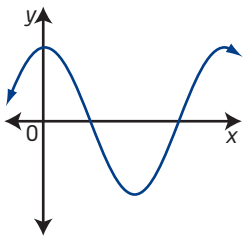
k) a maximum value occurs at (360°, 1)

l) a maximum value occurs at (90°, 1)

m)



n)



C5 **a)** Sketch the graph of $y = |\cos x|$ for $-2\pi \le x \le 2\pi$. How does the graph compare to the graph of $y = \cos x$?

b) Sketch the graph of $y = |\sin x|$ for $-2\pi \le x \le 2\pi$. How does the graph compare to the graph of $y = \sin x$?

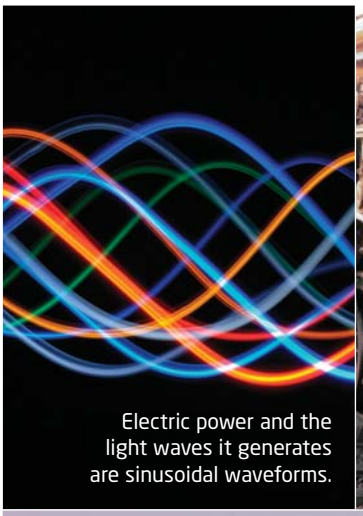
5.2

Transformations of Sinusoidal Functions

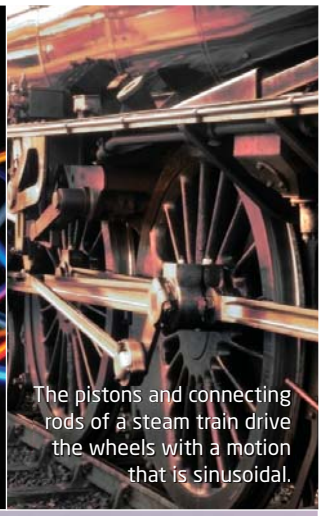
Focus on...

- graphing and transforming sinusoidal functions
- identifying the domain, range, phase shift, period, amplitude, and vertical displacement of sinusoidal functions
- developing equations of sinusoidal functions, expressed in radian and degree measure, from graphs and descriptions
- solving problems graphically that can be modelled using sinusoidal functions
- recognizing that more than one equation can be used to represent the graph of a sinusoidal function

The motion of a body attached to a suspended spring, the motion of the plucked string of a musical instrument, and the pendulum of a clock produce oscillatory motion that you can model with sinusoidal functions. To use the functions $y = \sin x$ and $y = \cos x$ in applied situations, such as these and the ones in the images shown, you need to be able to transform the functions.



Electric power and the light waves it generates are sinusoidal waveforms.



The pistons and connecting rods of a steam train drive the wheels with a motion that is sinusoidal.

Ocean waves created by the winds may be modelled by sinusoidal curves.

Investigate Transformations of Sinusoidal Functions

Materials

- grid paper
- graphing technology

A: Graph $y = \sin \theta + d$ or $y = \cos \theta + d$

1. On the same set of axes, sketch the graphs of the following functions for $0° \leq \theta \leq 360°$.

   $y = \sin \theta$
   $y = \sin \theta + 1$
   $y = \sin \theta - 2$

2. Using the language of transformations, compare the graphs of $y = \sin \theta + 1$ and $y = \sin \theta - 2$ to the graph of $y = \sin \theta$.
3. Predict what the graphs of $y = \sin \theta + 3$ and $y = \sin \theta - 4$ will look like. Justify your predictions.

Reflect and Respond

4. a) What effect does the parameter d in the function $y = \sin\theta + d$ have on the graph of $y = \sin\theta$ when $d > 0$?

b) What effect does the parameter d in the function $y = \sin\theta + d$ have on the graph of $y = \sin\theta$ when $d < 0$?

5. a) Predict the effect varying the parameter d in the function $y = \cos\theta + d$ has on the graph of $y = \cos\theta$.

b) Use a graph to verify your prediction.

B: Graph $y = \cos(\theta - c)$ or $y = \sin(\theta - c)$ Using Technology

6. On the same set of axes, sketch the graphs of the following functions for $-\pi \leq \theta \leq 2\pi$.

$y = \cos\theta$

$y = \cos\left(\theta + \frac{\pi}{2}\right)$

$y = \cos(\theta - \pi)$

7. Using the language of transformations, compare the graphs of $y = \cos\left(\theta + \frac{\pi}{2}\right)$ and $y = \cos(\theta - \pi)$ to the graph of $y = \cos\theta$.

8. Predict what the graphs of $y = \cos\left(\theta - \frac{\pi}{2}\right)$ and $y = \cos\left(\theta + \frac{3\pi}{2}\right)$ will look like. Justify your predictions.

Reflect and Respond

9. a) What effect does the parameter c in the function $y = \cos(\theta - c)$ have on the graph of $y = \cos\theta$ when $c > 0$?

b) What effect does the parameter c in the function $y = \cos(\theta - c)$ have on the graph of $y = \cos\theta$ when $c < 0$?

10. a) Predict the effect varying the parameter c in the function $y = \sin(\theta - c)$ has on the graph of $y = \sin\theta$.

b) Use a graph to verify your prediction.

Link the Ideas

You can translate graphs of functions up or down or left or right and stretch them vertically and/or horizontally. The rules that you have applied to the transformations of functions also apply to transformations of sinusoidal curves.

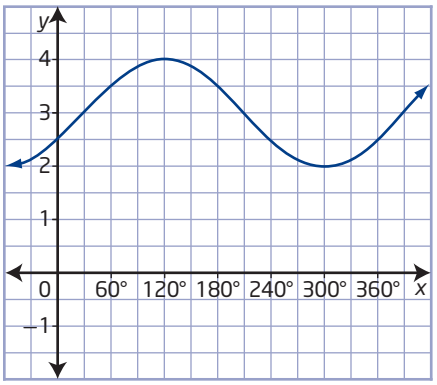
Example 1

Graph $y = \sin(x - c) + d$

a) Sketch the graph of the function $y = \sin(x - 30°) + 3$.
b) What are the domain and range of the function?
c) Use the language of transformations to compare your graph to the graph of $y = \sin x$.

Solution

a)



b) Domain: $\{x \mid x \in \mathrm{R}\}$
Range: $\{y \mid 2 \leq y \leq 4, y \in \mathrm{R}\}$

c) The graph has been translated 3 units up. This is the **vertical displacement**. The graph has also been translated 30° to the right. This is called the **phase shift**.

vertical displacement
- the vertical translation of the graph of a periodic function

phase shift
- the horizontal translation of the graph of a periodic function

Your Turn

a) Sketch the graph of the function $y = \cos(x + 45°) - 2$.
b) What are the domain and range of the function?
c) Use the language of transformations to compare your graph to the graph of $y = \cos x$.

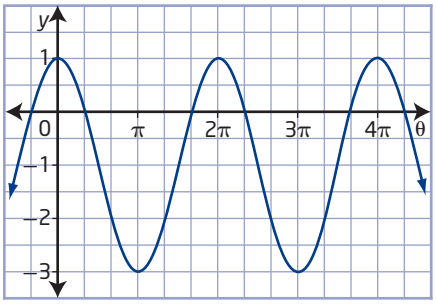
Example 2

Graph $y = a \cos(\theta - c) + d$

a) Sketch the graph of the function $y = -2 \cos(\theta + \pi) - 1$ over two cycles.
b) Use the language of transformations to compare your graph to the graph of $y = \cos \theta$. Indicate which parameter is related to each transformation.

Solution

a)



b) Since a is -2, the graph has been reflected about the θ-axis and then stretched vertically by a factor of two. The d-value is -1, so the graph is translated 1 unit down. The sinusoidal axis is defined as $y = -1$. Finally, the c-value is $-\pi$. Therefore, the graph is translated π units to the left.

Did You Know?

In this chapter, the parameters for horizontal and vertical translations are represented by c and d, respectively.

Your Turn

a) Sketch the graph of the function $y = 2 \sin\left(\theta - \frac{\pi}{2}\right) + 2$ over two cycles.

b) Compare your graph to the graph of $y = \sin \theta$.

Example 3

Graph $y = a \sin b(x - c) + d$

Sketch the graph of the function $y = 3 \sin\left(2x - \frac{2\pi}{3}\right) + 2$ over two cycles. What are the vertical displacement, amplitude, period, phase shift, domain, and range for the function?

Solution

First, rewrite the function in the standard form $y = a \sin b(x - c) + d$.

$y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$

Method 1: Graph Using Transformations

Step 1: Sketch the graph of $y = \sin x$ for one cycle. Apply the horizontal and vertical stretches to obtain the graph of $y = 3 \sin 2x$.

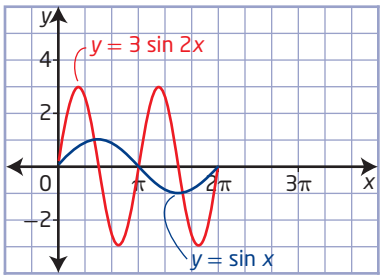
Compared to the graph of $y = \sin x$, the graph of $y = 3 \sin 2x$ is a horizontal stretch by a factor of $\frac{1}{2}$ and a vertical stretch by a factor of 3.

For the function $y = 3 \sin 2x$, $b = 2$.

$$\text{Period} = \frac{2\pi}{|b|}$$
$$= \frac{2\pi}{2}$$
$$= \pi$$

So, the period is π.

For the function $y = 3 \sin 2x$, $|a| = 3$.
So, the amplitude is 3.

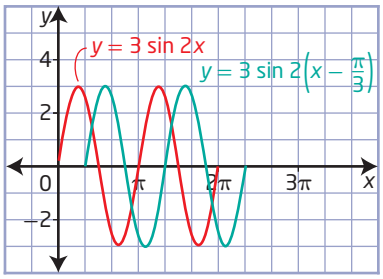


Step 2: Apply the horizontal translation to obtain the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right)$.

The phase shift is determined by the value of parameter c for a function in the standard form $y = a \sin b(x - c) + d$.

Compared to the graph of $y = 3 \sin 2x$, the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right)$ is translated horizontally $\frac{\pi}{3}$ units to the right.

The phase shift is $\frac{\pi}{3}$ units to the right.

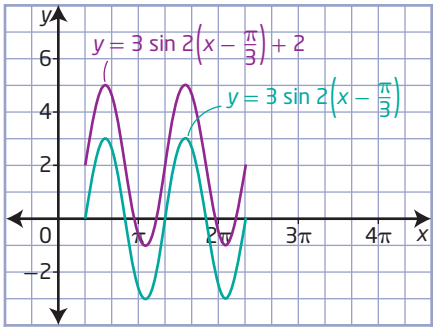


Step 3: Apply the vertical translation to obtain the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$.

The vertical displacement is determined by the value of parameter d for a function in the standard form $y = a \sin b(x - c) + d$.

Compared to the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right)$, the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$ is translated up 2 units.

The vertical displacement is 2 units up.



Would it matter if the order of the transformations were changed? Try a different order for the transformations.

Compared to the graph of $y = \sin x$, the graph of $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$ is

- horizontally stretched by a factor of $\frac{1}{2}$
- vertically stretched by a factor of 3
- horizontally translated $\frac{\pi}{3}$ units to the right
- vertically translated 2 units up

The vertical displacement is 2 units up.
The amplitude is 3.
The phase shift is $\frac{\pi}{3}$ units to the right.
The domain is $\{x \mid x \in \mathrm{R}\}$.
The range is $\{y \mid -1 \leq y \leq 5, y \in \mathrm{R}\}$.

Method 2: Graph Using Key Points

You can identify five key points to graph one cycle of the sine function. The first, third, and fifth points indicate the start, the middle, and the end of the cycle. The second and fourth points indicate the maximum and minimum points.

Comparing $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$ to $y = a \sin b(x - c) + d$ gives $a = 3$, $b = 2$, $c = \frac{\pi}{3}$, and $d = 2$.

The amplitude is $|a|$, or 3.

The period is $\frac{2\pi}{|b|}$, or π.

The vertical displacement is d, or 2. Therefore, the equation of the sinusoidal axis or mid-line is $y = 2$.

You can use the amplitude and vertical displacement to determine the maximum and minimum values.

The maximum value is
$d + |a| = 2 + 3$
$= 5$

The minimum value is
$d - |a| = 2 - 3$
$= -1$

Determine the values of x for the start and end of one cycle from the function $y = a \sin b(x - c) + d$ by solving the compound inequality $0 \leq b(x - c) \leq 2\pi$.

$0 \leq 2\left(x - \frac{\pi}{3}\right) \leq 2\pi$

How does this inequality relate to the period of the function?

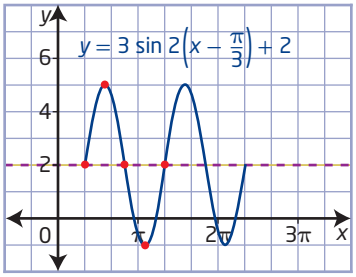
$0 \leq x - \frac{\pi}{3} \leq \pi$

$\frac{\pi}{3} \leq x \leq \frac{4\pi}{3}$

Divide the interval $\frac{\pi}{3} \le x \le \frac{4\pi}{3}$ into four equal segments. By doing this, you can locate five key values of x along the sinusoidal axis.

$\frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}$

Use the above information to sketch one cycle of the graph, and then a second cycle.



Note the five key points and how you can use them to sketch one cycle of the graph of the function.

For the graph of the function $y = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 2$,

- the vertical displacement is 2 units up
- the amplitude is 3
- the phase shift is $\frac{\pi}{3}$ units to the right
- the domain is $\{x \mid x \in \mathrm{R}\}$
- the range is $\{y \mid -1 \le y \le 5, y \in \mathrm{R}\}$

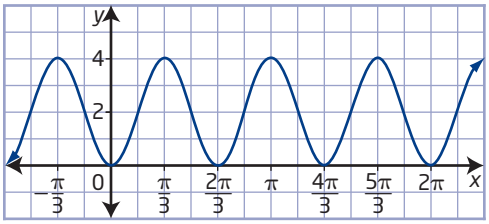
Your Turn

Sketch the graph of the function $y = 2 \cos 4(x + \pi) - 1$ over two cycles. What are the vertical displacement, amplitude, period, phase shift, domain, and range for the function?

Example 4

Determine an Equation From a Graph

The graph shows the function $y = f(x)$.



a) Write the equation of the function in the form $y = a \sin b(x - c) + d$, $a > 0$.

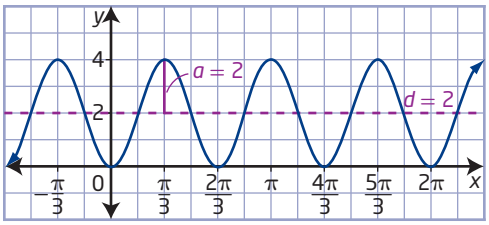
b) Write the equation of the function in the form $y = a \cos b(x - c) + d$, $a > 0$.

c) Use technology to verify your solutions.

Solution

a) Determine the values of the parameters a, b, c, and d.

Locate the sinusoidal axis or mid-line. Its position determines the value of d. Thus, $d = 2$.



Use the sinusoidal axis from the graph or use the formula to determine the amplitude.

How can you use the maximum and minimum values of the graph to find the value of d?

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$a = \frac{4 - 0}{2}$$

$$a = 2$$

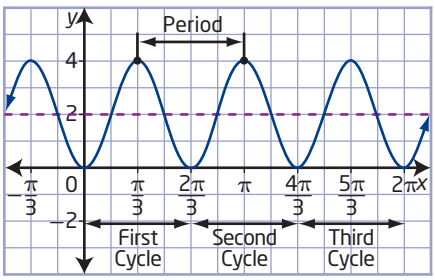
The amplitude is 2.

Determine the period and the value of b.

Method 1: Count the Number of Cycles in 2π

Determine the number of cycles in a distance of 2π.

In this function, there are three cycles. Therefore, the value of b is 3 and the period is $\frac{2\pi}{3}$.



Method 2: Determine the Period First

Locate the start and end of one cycle of the sine curve.

Recall that one cycle of $y = \sin x$ starts at (0, 0). How is that point transformed? How could this information help you determine the start for one cycle of this sine curve?

The start of the first cycle of the sine curve that is closest to the y-axis is at $x = \frac{\pi}{6}$ and the end is at $x = \frac{5\pi}{6}$.

The period is $\frac{5\pi}{6} - \frac{\pi}{6}$, or $\frac{2\pi}{3}$.

Solve the equation for b.

$$\text{Period} = \frac{2\pi}{|b|}$$

$$\frac{2\pi}{3} = \frac{2\pi}{|b|}$$

$$b = 3$$

Choose b to be positive.

Determine the phase shift, c.

Locate the start of the first cycle of the sine curve to the right of the y-axis. Thus, $c = \frac{\pi}{6}$.

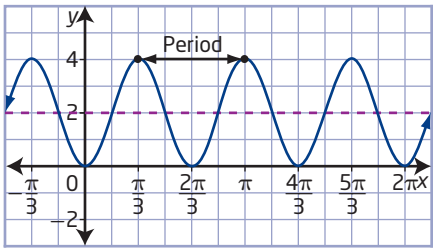
Substitute the values of the parameters $a = 2$, $b = 3$, $c = \frac{\pi}{6}$, and $d = 2$ into the equation $y = a \sin b(x - c) + d$.

The equation of the function in the form $y = a \sin b(x - c) + d$ is $y = 2 \sin 3\left(x - \frac{\pi}{6}\right) + 2$.

b) To write an equation in the form $y = a \cos b(x - c) + d$, determine the values of the parameters a, b, c, and d using steps similar to what you did for the sine function in part a).

$a = 2$
$b = 3$
$c = \frac{\pi}{3}$
$d = 2$

Why is $c = \frac{\pi}{3}$? Are there other possible values for c?

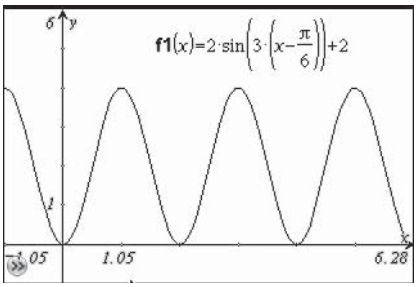


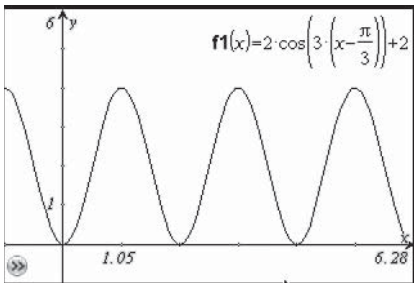
The equation of the function in the form $y = a \cos b(x - c) + d$ is $y = 2 \cos 3\left(x - \frac{\pi}{3}\right) + 2$.

How do the two equations compare?

Could other equations define the function $y = f(x)$?

c) Enter the functions on a graphing calculator. Compare the graphs to the original and to each other.

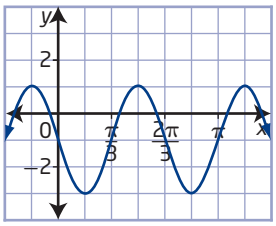




The graphs confirm that the equations for the function are correct.

Your Turn

The graph shows the function $y = f(x)$.



a) Write the equation of the function in the form $y = a \sin b(x - c) + d$, $a > 0$.
b) Write the equation of the function in the form $y = a \cos b(x - c) + d$, $a > 0$.
c) Use technology to verify your solutions.

Example 5

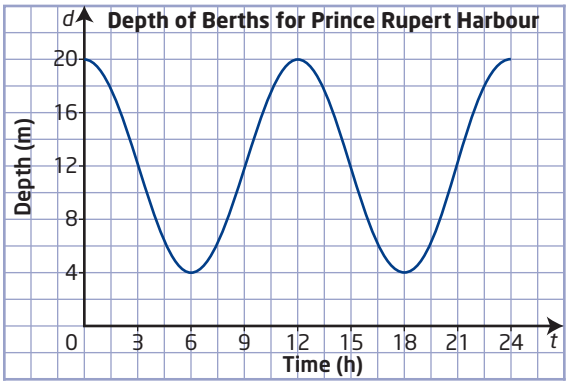
Interpret Graphs of Sinusoidal Functions

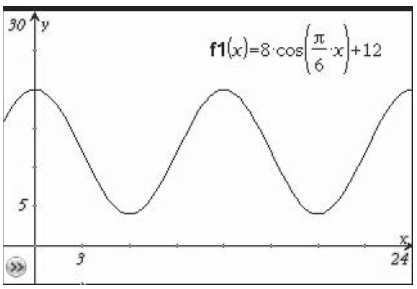
Prince Rupert, British Columbia, has the deepest natural harbour in North America. The depth, d, in metres, of the berths for the ships can be approximated by the equation $d(t) = 8 \cos \frac{\pi}{6}t + 12$, where t is the time, in hours, after the first high tide.

a) Graph the function for two cycles.
b) What is the period of the tide?
c) An ocean liner requires a minimum of 13 m of water to dock safely. From the graph, determine the number of hours per cycle the ocean liner can safely dock.
d) If the minimum depth of the berth occurs at 6 h, determine the depth of the water. At what other times is the water level at a minimum? Explain your solution.

Solution

a)





Why should you set the calculator to radian mode when graphing sinusoidal functions that represent real-world situations?

b) Use $b = \frac{\pi}{6}$ to determine the period.

$$\text{Period} = \frac{2\pi}{|b|}$$

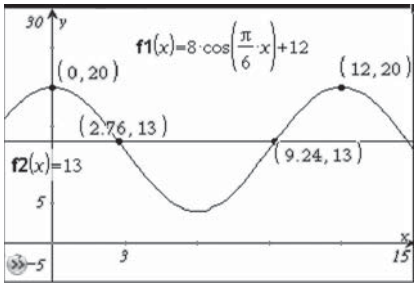
$$\text{Period} = \frac{2\pi}{\left|\frac{\pi}{6}\right|}$$

$$\text{Period} = 12$$

The period for the tides is 12 h.

What does the period of 12 h represent?

c) To determine the number of hours an ocean liner can dock safely, draw the line $y = 13$ to represent the minimum depth of the berth. Determine the points of intersection of the graphs of $y = 13$ and $d(t) = 8 \cos \frac{\pi}{6}t + 12$.



More precise answers can be obtained using technology.

The points of intersection for the first cycle are approximately (2.76, 13) and (9.26, 13).

The depth is greater than 13 m from 0 h to approximately 2.76 h and from approximately 9.24 h to 12 h. The total time when the depth is greater than 13 m is 2.76 + 2.76, or 5.52 h, or about 5 h 30 min per cycle.

d) To determine the berth depth at 6 h, substitute the value of $t = 6$ into the equation.

$d(t) = 8 \cos \frac{\pi}{6}t + 12$

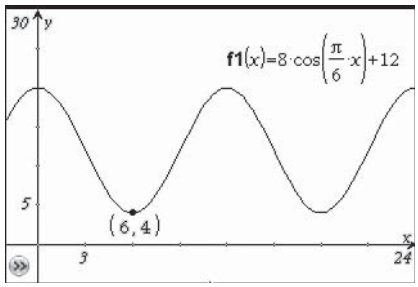
$d(6) = 8 \cos \frac{\pi}{6}(6) + 12$

$d(6) = 8 \cos \pi + 12$

$d(6) = 8(-1) + 12$

$d(6) = 4$

You can use the graph to verify the solution.



The berth depth at 6 h is 4 m. Add 12 h (the period) to 6 h to determine the next time the berth depth is 4 m. Therefore, the berth depth of 4 m occurs again at 18 h.

Your Turn

The depth, d, in metres, of the water in the harbour at New Westminster, British Columbia, is approximated by the equation $d(t) = 0.6 \cos \frac{2\pi}{13}t + 3.7$, where t is the time, in hours, after the first high tide.

a) Graph the function for two cycles starting at $t = 0$.

b) What is the period of the tide?

c) If a boat requires a minimum of 3.5 m of water to launch safely, for how many hours per cycle can the boat safely launch?

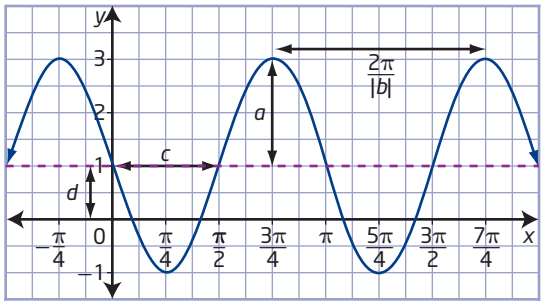
d) What is the depth of the water at 7 h? At what other times is the water level at this depth? Explain your solution.

Key Ideas

- You can determine the amplitude, period, phase shift, and vertical displacement of sinusoidal functions when the equation of the function is given in the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$.

For: $y = a \sin b(x - c) + d$
$y = a \cos b(x - c) + d$

How does changing each parameter affect the graph of a function?



Vertical stretch by a factor of $|a|$
- changes the amplitude to $|a|$
- reflected in the x-axis if $a < 0$

Horizontal stretch by a factor of $\frac{1}{|b|}$
- changes the period to $\frac{360°}{|b|}$ (in degrees) or $\frac{2\pi}{|b|}$ (in radians)
- reflected in the y-axis if $b < 0$

Horizontal phase shift represented by c
- to right if $c > 0$
- to left if $c < 0$

Vertical displacement represented by d
- up if $d > 0$
- down if $d < 0$

$$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$

- You can determine the equation of a sinusoidal function given its properties or its graph.

Check Your Understanding

Practise

1. Determine the phase shift and the vertical displacement with respect to $y = \sin x$ for each function. Sketch a graph of each function.

a) $y = \sin(x - 50°) + 3$

b) $y = \sin(x + \pi)$

c) $y = \sin\left(x + \frac{2\pi}{3}\right) + 5$

d) $y = 2\sin(x + 50°) - 10$

e) $y = -3\sin(6x + 30°) - 3$

f) $y = 3\sin\frac{1}{2}\left(x - \frac{\pi}{4}\right) - 10$

2. Determine the phase shift and the vertical displacement with respect to $y = \cos x$ for each function. Sketch a graph of each function.

a) $y = \cos(x - 30°) + 12$

b) $y = \cos\left(x - \frac{\pi}{3}\right)$

c) $y = \cos\left(x + \frac{5\pi}{6}\right) + 16$

d) $y = 4\cos(x + 15°) + 3$

e) $y = 4\cos(x - \pi) + 4$

f) $y = 3\cos\left(2x - \frac{\pi}{6}\right) + 7$

3. a) Determine the range of each function.

i) $y = 3\cos\left(x - \frac{\pi}{2}\right) + 5$

ii) $y = -2\sin(x + \pi) - 3$

iii) $y = 1.5\sin x + 4$

iv) $y = \frac{2}{3}\cos(x + 50°) + \frac{3}{4}$

b) Describe how to determine the range when given a function of the form $y = a\cos b(x - c) + d$ or $y = a\sin b(x - c) + d$.

4. Match each function with its description in the table.

a) $y = -2\cos 2(x + 4) - 1$

b) $y = 2\sin 2(x - 4) - 1$

c) $y = 2\sin(2x - 4) - 1$

d) $y = 3\sin(3x - 9) - 1$

e) $y = 3\sin(3x + \pi) - 1$

	Amplitude	Period	Phase Shift	Vertical Displacement
A	3	$\frac{2\pi}{3}$	3 right	1 down
B	2	π	2 right	1 down
C	2	π	4 right	1 down
D	2	π	4 left	1 down
E	3	$\frac{2\pi}{3}$	$\frac{\pi}{3}$ left	1 down

5. Match each function with its graph.

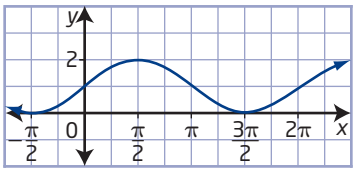
a) $y = \sin\left(x - \frac{\pi}{4}\right)$

b) $y = \sin\left(x + \frac{\pi}{4}\right)$

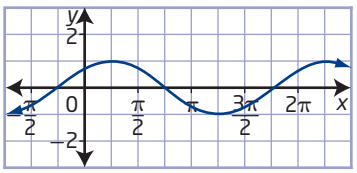
c) $y = \sin x - 1$

d) $y = \sin x + 1$

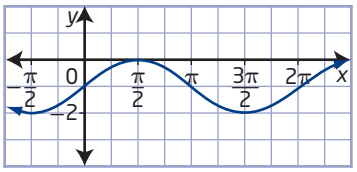
A



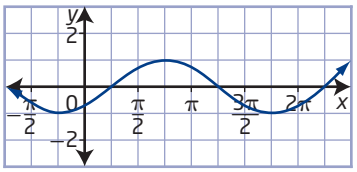
B



C



D



Apply

6. Write the equation of the sine function in the form $y = a \sin b(x - c) + d$ given its characteristics.

a) amplitude 4, period π, phase shift $\frac{\pi}{2}$ to the right, vertical displacement 6 units down

b) amplitude 0.5, period 4π, phase shift $\frac{\pi}{6}$ to the left, vertical displacement 1 unit up

c) amplitude $\frac{3}{4}$, period 720°, no phase shift, vertical displacement 5 units down

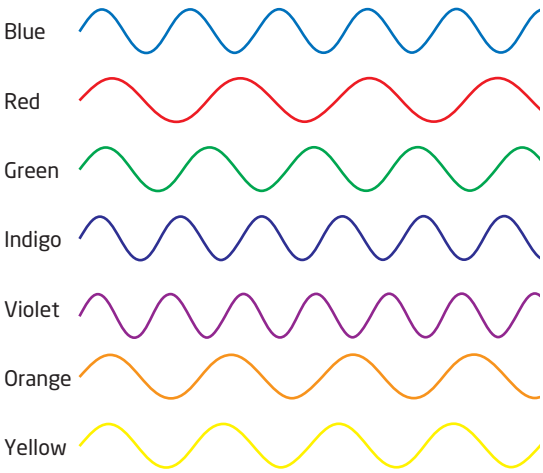
7. The graph of $y = \cos x$ is transformed as described. Determine the values of the parameters a, b, c, and d for the transformed function. Write the equation for the transformed function in the form $y = a \cos b(x - c) + d$.

a) vertical stretch by a factor of 3 about the x-axis, horizontal stretch by a factor of 2 about the y-axis, translated 2 units to the left and 3 units up

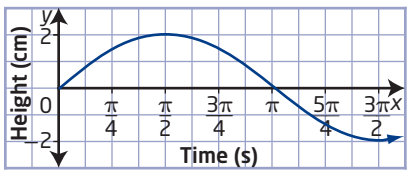
b) vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, horizontal stretch by a factor of $\frac{1}{4}$ about the y-axis, translated 3 units to the right and 5 units down

c) vertical stretch by a factor of $\frac{3}{2}$ about the x-axis, horizontal stretch by a factor of 3 about the y-axis, reflected in the x-axis, translated $\frac{\pi}{4}$ units to the right and 1 unit down

8. When white light shines through a prism, the white light is broken into the colours of the visible light spectrum. Each colour corresponds to a different wavelength of the electromagnetic spectrum. Arrange the colours, in order from greatest to smallest period.



9. The piston engine is the most commonly used engine in the world. The height of the piston over time can be modelled by a sine curve. Given the equation for a sine curve, $y = a \sin b(x - c) + d$, which parameter(s) would be affected as the piston moves faster?



10. Victor and Stewart determined the phase shift for the function $f(x) = 4 \sin (2x - 6) + 12$. Victor said that the phase shift was 6 units to the right, while Stewart claimed it was 3 units to the right.

a) Which student was correct? Explain your reasoning.

b) Graph the function to verify your answer from part a).

11. A family of sinusoidal graphs with equations of the form $y = a \sin b(x - c) + d$ is created by changing only the vertical displacement of the function. If the range of the original function is $\{y \mid -3 \leq y \leq 3, y \in \mathrm{R}\}$, determine the range of the function with each given value of d.

a) $d = 2$

b) $d = -3$

c) $d = -10$

d) $d = 8$

12. Sketch the graph of the curve that results after applying each transformation to the graph of the function $f(x) = \sin x$.

a) $f\left(x - \frac{\pi}{3}\right)$

b) $f\left(x + \frac{\pi}{4}\right)$

c) $f(x) + 3$

d) $f(x) - 4$

13. The range of a trigonometric function in the form $y = a \sin b(x - c) + d$ is $\{y \mid -13 \leq y \leq 5, y \in \mathrm{R}\}$. State the values of a and d.

14. For each graph of a sinusoidal function, state

i) the amplitude

ii) the period

iii) the phase shift

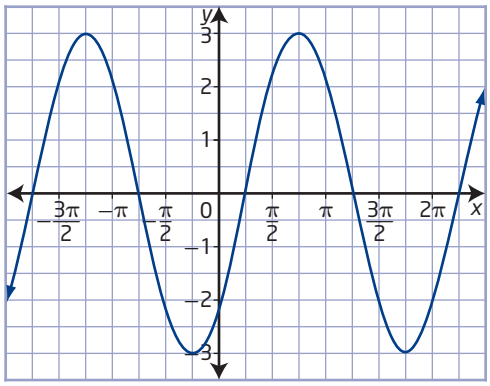
iv) the vertical displacement

v) the domain and range

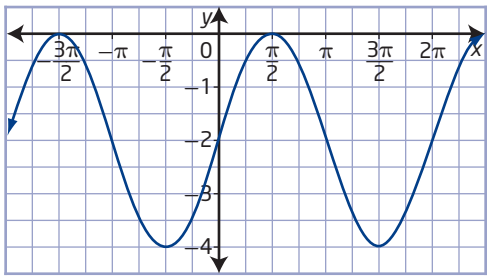
vi) the maximum value of y and the values of x for which it occurs over the interval $0 \leq x \leq 2\pi$

vii) the minimum value of y and the values of x for which it occurs over the interval $0 \leq x \leq 2\pi$

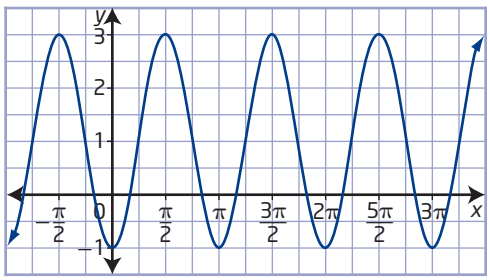
a) a sine function



b) a cosine function

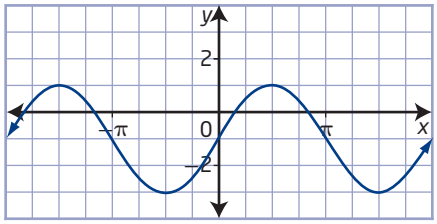


c) a sine function



15. Determine an equation in the form $y = a \sin b(x - c) + d$ for each graph.

a)



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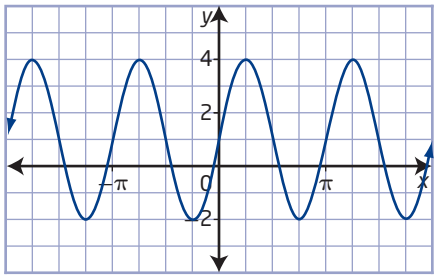
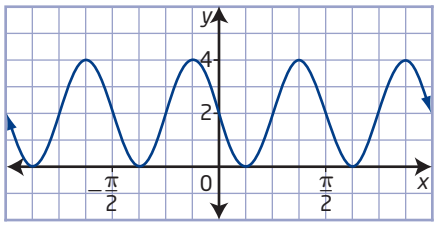
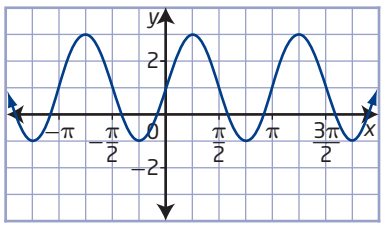


c)



16. For each graph, write an equation in the form $y = a \cos b(x - c) + d$.

a)



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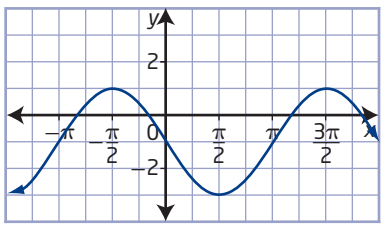
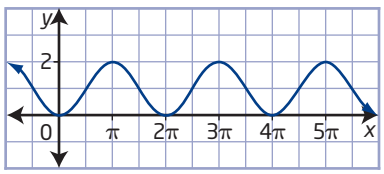


c)



17. a) Graph the function $f(x) = \cos\left(x - \frac{\pi}{2}\right)$.

b) Consider the graph. Write an equation of the function in the form $y = a \sin b(x - c) + d$.

c) What conclusions can you make about the relationship between the two equations of the function?

18. Given the graph of the function $f(x) = \sin x$, what transformation is required so that the function $g(x) = \cos x$ describes the graph of the image function?

19. For each start and end of one cycle of a cosine function in the form $y = 3 \cos b(x - c)$,

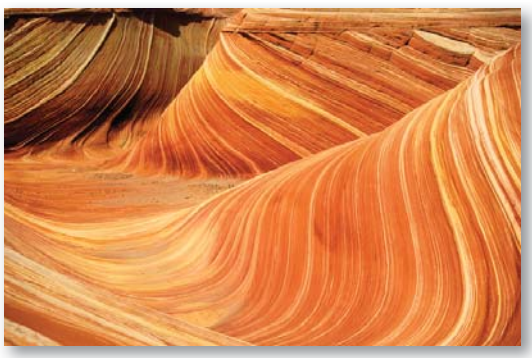
i) state the phase shift, period, and x-intercepts

ii) state the coordinates of the minimum and maximum values

a) $30° \le x \le 390°$

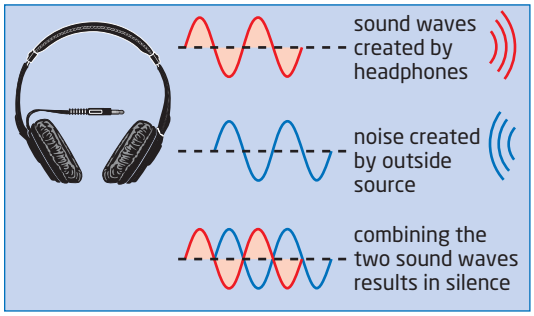
b) $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

20. The Wave is a spectacular sandstone formation on the slopes of the Coyote Buttes of the Paria Canyon in Northern Arizona. The Wave is made from 190 million-year-old sand dunes that have turned to red rock. Assume that a cycle of the Wave may be approximated using a cosine curve. The maximum height above sea level is 5100 ft and the minimum height is 5000 ft. The beginning of the cycle is at the 1.75 mile mark of the canyon and the end of this cycle is at the 2.75 mile mark. Write an equation that approximates the pattern of the Wave.



21. Compare the graphs of the functions $y = 3 \sin \frac{\pi}{3}(x - 2) - 1$ and $y = 3 \cos \frac{\pi}{3}\left(x - \frac{7}{2}\right) - 1$. Are the graphs equivalent? Support your answer graphically.

22. Noise-cancelling headphones are designed to give you maximum listening pleasure by cancelling ambient noise and actively creating their own sound waves. These waves mimic the incoming noise in every way, except that they are out of sync with the intruding noise by 180°.



Suppose that the amplitude and period for the sine waves created by the outside noise are 4 and $\frac{\pi}{2}$, respectively. Determine the equation of the sound waves the headphones produce to effectively cancel the ambient noise.

23. The overhang of the roof of a house is designed to shade the windows for cooling in the summer and allow the Sun's rays to enter the house for heating in the winter. The Sun's angle of elevation, A, in degrees, at noon in Estevan, Saskatchewan, can be modelled by the formula $A = -23.5 \sin \frac{360}{365}(x + 102) + 41$, where x is the number of days elapsed beginning with January 1.

a) Use technology to sketch the graph showing the changes in the Sun's angle of elevation throughout the year.

b) Determine the Sun's angle of elevation at noon on February 12.

c) On what date is the angle of elevation the greatest in Estevan?

24. After exercising for 5 min, a person has a respiratory cycle for which the rate of air flow, r, in litres per second, in the lungs is approximated by $r = 1.75 \sin \frac{\pi}{2}t$, where t is the time, in seconds.

a) Determine the time for one full respiratory cycle.

b) Determine the number of cycles per minute.

c) Sketch the graph of the rate of air flow function.

d) Determine the rate of air flow at a time of 30 s. Interpret this answer in the context of the respiratory cycle.

e) Determine the rate of air flow at a time of 7.5 s. Interpret this answer in the context of the respiratory cycle.

Extend

25. The frequency of a wave is the number of cycles that occur in 1 s. Adding two sinusoidal functions with similar, but unequal, frequencies results in a function that pulsates, or exhibits beats. Piano tuners often use this phenomenon to help them tune a piano.

a) Graph the function $y = \cos x + \cos 0.9x$.

b) Determine the amplitude and the period of the resulting wave.

26. a) Copy each equation. Fill in the missing values to make the equation true.

i) $4 \sin (x - 30°) = 4 \cos (x - \blacksquare)$

ii) $2 \sin \left(x - \frac{\pi}{4}\right) = 2 \cos (x - \blacksquare)$

iii) $-3 \cos \left(x - \frac{\pi}{2}\right) = 3 \sin (x + \blacksquare)$

iv) $\cos (-2x + 6\pi) = \sin 2(x + \blacksquare)$

b) Choose one of the equations in part a) and explain how you got your answer.

27. Determine the equation of the sine function with

a) amplitude 3, maximum $\left(-\frac{\pi}{2}, 5\right)$, and nearest maximum to the right at $\left(\frac{3\pi}{2}, 5\right)$

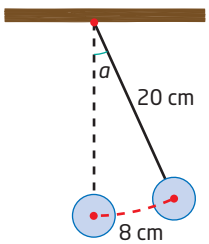
b) amplitude 3, minimum $\left(\frac{\pi}{4}, -2\right)$, and nearest maximum to the right at $\left(\frac{3\pi}{4}, 4\right)$

c) minimum $(-\pi, 3)$ and nearest maximum to the right at $(0, 7)$

d) minimum $(90°, -6)$ and nearest maximum to the right at $(150°, 4)$

28. The angle, P, in radians, between a pendulum and the vertical may be modelled by the equation $P = a \cos bt$, where a represents the maximum angle that the pendulum swings from the vertical; b is the horizontal stretch factor; and t is time, in seconds. The period of a pendulum may be approximated by the formula Period $= 2\pi\sqrt{\frac{L}{g}}$, where L is the pendulum length and g is the acceleration due to gravity (9.8 m/s^2).

a) Sketch the graph that models the position of the pendulum in the diagram from $0 \leq t \leq 5$.



b) Determine the position of the pendulum after 6 s. Express your answer to the nearest tenth of a centimetre.

Create Connections

C1 Consider a sinusoidal function of the form $y = a \sin b(x - c) + d$. Describe the effect that each of the parameters a, b, c, and d has on the graph of the function. Compare this to what you learned in Chapter 1 Function Transformations.

C2 Sketch the graphs of $y = -\sin x$ and $y = \sin(-x)$.

a) Compare the two graphs. How are they alike? different?

b) Explain why this happens.

c) How would you expect the graphs of $y = -\cos x$ and $y = \cos(-x)$ to compare?

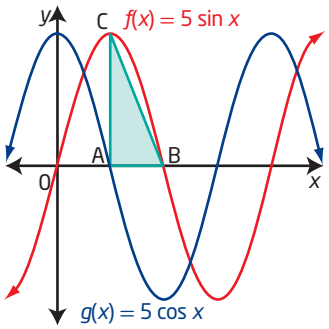
d) Check your hypothesis from part c). If it is incorrect, write a correct statement about the cosine function.

Did You Know?

An *even function* satisfies the property $f(-x) = f(x)$ for all x in the domain of $f(x)$.

An *odd function* satisfies the property $f(-x) = -f(x)$ for all x in the domain of $f(x)$.

C3 Triangle ABC is inscribed between the graphs of $f(x) = 5 \sin x$ and $g(x) = 5 \cos x$. Determine the area of $\triangle ABC$.



C4 The equation of a sine function can be expressed in the form $y = a \sin b(x - c) + d$. Determine the values of the parameters a, b, c, and/or d, where $a > 0$ and $b > 0$, for each of the following to be true.

a) The period is greater than 2π.

b) The amplitude is greater than 1 unit.

c) The graph passes through the origin.

d) The graph has no x-intercepts.

e) The graph has a y-intercept of a.

f) The length of one cycle is 120°.

5.3

The Tangent Function

Focus on...

- sketching the graph of $y = \tan x$
- determining the amplitude, domain, range, and period of $y = \tan x$
- determining the asymptotes and x-intercepts for the graph of $y = \tan x$
- solving a problem by analysing the graph of the tangent function

You can derive the tangent of an angle from the coordinates of a point on a line tangent to the unit circle at point (1, 0). These values have been tabulated and programmed into scientific calculators and computers. This allows you to apply trigonometry to surveying, engineering, and navigation problems.

Did You Know?

Tangent comes from the Latin word *tangere*, "to touch."

Tangent was first mentioned in 1583 by T. Fincke, who introduced the word *tangens* in Latin. E. Gunter (1624) used the notation *tan*, and J.H. Lambert (1770) discovered the fractional representation of this function.

Investigate the Tangent Function

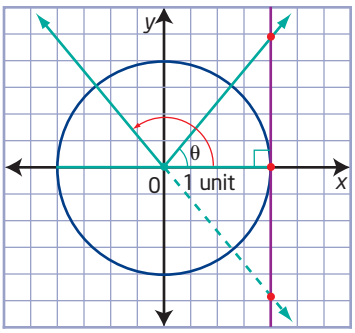
Materials

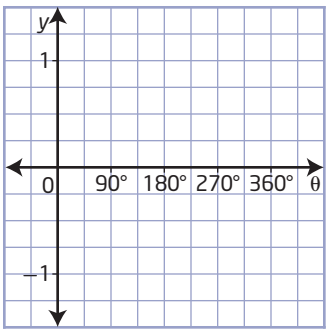
- grid paper
- ruler
- protractor
- compass
- graphing technology

A: Graph the Tangent Function

A tangent line to a curve is a line that touches a curve, or a graph of a function, at a single point.

1. On a piece of grid paper, draw and label the x-axis and y-axis. Draw a circle of radius 1 so that its centre is at the origin. Draw a tangent to the circle at the point where the x-axis intersects the circle on the right side.
2. To sketch the graph of the tangent function over the interval $0° \leq \theta \leq 360°$, you can draw angles in standard position on the unit circle and extend the terminal arm to the right so that it intersects the tangent line, as shown in the diagram. The y-coordinate of the point of intersection represents the value of the tangent function. Plot points represented by the coordinates (angle measure, y-coordinate of point of intersection).





a) Begin with an angle of 0°. Where does the extension of the terminal arm intersect the tangent line?

b) Draw the terminal arm for an angle of 45°. Where does the extension of the terminal arm intersect the tangent line?

c) If the angle is 90°, where does the extension of the terminal arm intersect the tangent line?

What can you conclude about the value of tan 90°? How do you show this on a graph?

d) Use a protractor to measure various angles for the terminal arm. Determine the *y*-coordinate of the point where the terminal arm intersects the tangent line. Plot the ordered pair (angle measure, *y*-coordinate on tangent line) on a graph like the one shown above on the right.

Angle Measure	0°	45°	90°	135°	180°	225°	270°	315°	360°
y-coordinate on Tangent Line									

3. Use graphing technology to verify the shape of your graph.

Reflect and Respond

4. When $\theta = 90°$ and $\theta = 270°$, the tangent function is undefined. How does this relate to the graph of the tangent function?

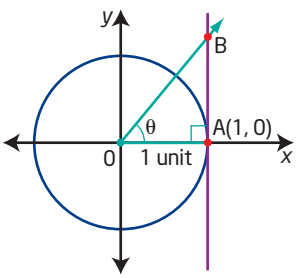
5. What is the period of the tangent function?

6. What is the amplitude of the tangent function? What does this mean?

7. Explain how a point P(x, y) on the unit circle relates to the sine, cosine, and tangent ratios.

B: Connect the Tangent Function to the Slope of the Terminal Arm

8. The diagram shows an angle θ in standard position whose terminal arm intersects the tangent AB at point B. Express the ratio of tan θ in terms of the sides of $\triangle AOB$.



9. Using your knowledge of special triangles, state the exact value of tan 60°. If θ = 60° in the diagram, what is the length of line segment AB?
10. Using the measurement of the length of line segment AB from step 9, determine the slope of line segment OB.
11. How does the slope of line segment OB relate to the tangent of an angle in standard position?

Reflect and Respond

12. How could you use the concept of slope to determine the tangent ratio when θ = 0°? when θ = 90°?
13. Using a calculator, determine the values of tan θ as θ approaches 90°. What is tan 90°?
14. Explain the relationship between the terminal arm of an angle θ and the tangent of the line passing through the point (1, 0) when θ = 90°. (Hint: Can the terminal arm intersect the tangent line?)

Link the Ideas

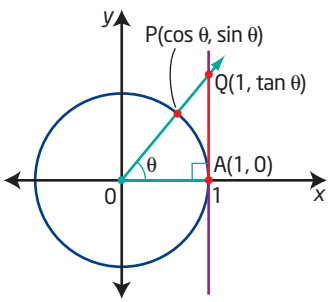
The value of the tangent of an angle θ is the slope of the line passing through the origin and the point on the unit circle (cos θ, sin θ). You can think of it as the slope of the terminal arm of angle θ in standard position.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

When sin θ = 0, what is tan θ? Explain.
When cos θ = 0, what is tan θ? Explain.

The tangent ratio is the length of the line segment tangent to the unit circle at the point A(1, 0) from the *x*-axis to the terminal arm of angle θ at point Q.

From the diagram, the distance AQ is equal to the *y*-coordinate of point Q. Therefore, point Q has coordinates (1, tan θ).



How could you show that the coordinates of Q are (1, tan θ)?

Example 1

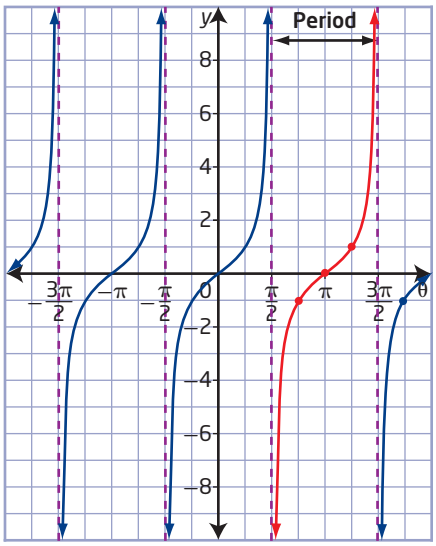
Graph the Tangent Function

Graph the function $y = \tan\theta$ for $-2\pi \leq \theta \leq 2\pi$. Describe its characteristics.

Solution

The function $y = \tan\theta$ is known as the tangent function. Using the unit circle, you can plot values of y against the corresponding values of θ.

Between asymptotes, the graph of $y = \tan\theta$ passes through a point with y-coordinate -1, a θ-intercept, and a point with y-coordinate 1.



You can observe the properties of the tangent function from the graph.

- The curve is not continuous. It breaks at $\theta = -\frac{3\pi}{2}$, $\theta = -\frac{\pi}{2}$, $\theta = \frac{\pi}{2}$, and $\theta = \frac{3\pi}{2}$, where the function is undefined.
- $\tan\theta = 0$ when $\theta = -2\pi$, $\theta = -\pi$, $\theta = 0$, $\theta = \pi$, and $\theta = 2\pi$.
- $\tan\theta = 1$ when $\theta = -\frac{7\pi}{4}$, $\theta = -\frac{3\pi}{4}$, $\theta = \frac{\pi}{4}$, and $\theta = \frac{5\pi}{4}$.
- $\tan\theta = -1$ when $\theta = -\frac{5\pi}{4}$, $\theta = -\frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$, and $\theta = \frac{7\pi}{4}$.
- The graph of $y = \tan\theta$ has no amplitude because it has no maximum or minimum values.
- The range of $y = \tan\theta$ is $\{y \mid y \in \mathrm{R}\}$.

- As point P moves around the unit circle in either a clockwise or a counterclockwise direction, the tangent curve repeats for every interval of π. The period for $y = \tan\theta$ is π.

For tangent graphs, the distance between any two consecutive vertical asymptotes represents one complete period.

- The tangent is undefined whenever $\cos\theta = 0$. This occurs when $\theta = \frac{\pi}{2} + n\pi$, $n \in \mathrm{I}$. At these points, the value of the tangent approaches infinity and is undefined. When graphing the tangent, use dashed lines to show where the value of the tangent is undefined. These vertical lines are called asymptotes.

Why is $\tan\theta$ undefined for $\cos\theta = 0$?

- The domain of $y = \tan\theta$ is $\left\{\theta \mid \theta \neq \frac{\pi}{2} + n\pi, \theta \in \mathrm{R}, n \in \mathrm{I}\right\}$.

Your Turn

Graph the function $y = \tan\theta$, $0° \leq \theta \leq 360°$. Describe how the characteristics are different from those in Example 1.

Example 2

Model a Problem Using the Tangent Function

A small plane is flying at a constant altitude of 6000 m directly toward an observer. Assume that the ground is flat in the region close to the observer.

Why is this assumption made?

a) Determine the relation between the horizontal distance, in metres, from the observer to the plane and the angle, in degrees, formed from the vertical to the plane.

b) Sketch the graph of the function.

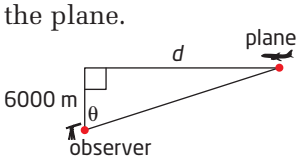
c) Where are the asymptotes located in this graph? What do they represent?

d) Explain what happens when the angle is equal to 0°.

Solution

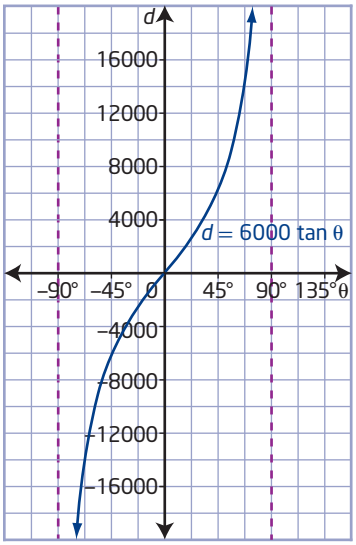
a) Draw a diagram to model the situation.

Let d represent the horizontal distance from the observer to the plane. Let θ represent the angle formed by the vertical and the line of sight to the plane.



$$\tan\theta = \frac{d}{6000}$$
$$d = 6000\tan\theta$$

b) The graph represents the horizontal distance between the plane and the observer. As the plane flies toward the observer, that distance decreases. As the plane moves from directly overhead to the observer's left, the distance values become negative. The domain of the function is $\{\theta \mid -90° < \theta < 90°, \theta \in R\}$.



c) The asymptotes are located at $\theta = 90°$ and $\theta = -90°$. They represent when the plane is on the ground to the right or left of the observer, which is impossible, because the plane is flying in a straight line at a constant altitude of 6000 m.

d) When the angle is equal to 0°, the plane is directly over the head of the observer. The horizontal distance is 0 m.

Your Turn

A small plane is flying at a constant altitude of 5000 m directly toward an observer. Assume the ground is flat in the region close to the observer.

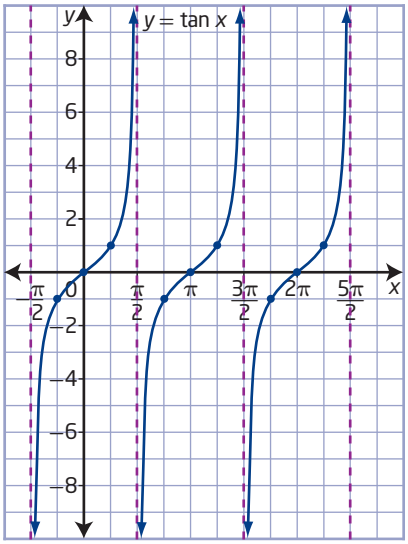
a) Sketch the graph of the function that represents the relation between the horizontal distance, in metres, from the observer to the plane and the angle, in degrees, formed by the vertical and the line of sight to the plane.

b) Use the characteristics of the tangent function to describe what happens to the graph as the plane flies from the right of the observer to the left of the observer.

Key Ideas

- You can use asymptotes and three points to sketch one cycle of a tangent function. To graph $y = \tan x$, draw one asymptote; draw the points where $y = -1$, $y = 0$, and $y = 1$; and then draw another asymptote.

How can you determine the location of the asymptotes for the function $y = \tan x$?



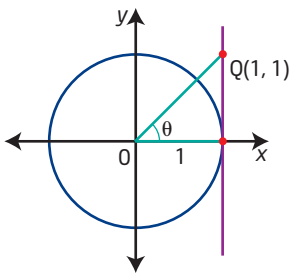
- The tangent function $y = \tan x$ has the following characteristics:
  - The period is π.
  - The graph has no maximum or minimum values.
  - The range is $\{y \mid y \in \mathrm{R}\}$.
  - Vertical asymptotes occur at $x = \frac{\pi}{2} + n\pi$, $n \in \mathrm{I}$.
  - The domain is $\left\{x \mid x \neq \frac{\pi}{2} + n\pi, x \in \mathrm{R}, n \in \mathrm{I}\right\}$.
  - The x-intercepts occur at $x = n\pi$, $n \in \mathrm{I}$.
  - The y-intercept is 0.

Check Your Understanding

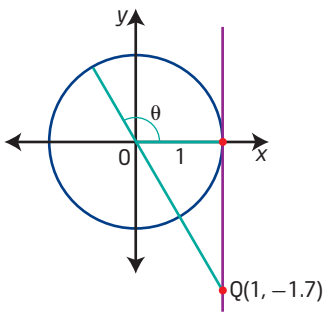
Practise

1. For each diagram, determine $\tan\theta$ and the value of θ, in degrees. Express your answer to the nearest tenth, when necessary.

a)



b)



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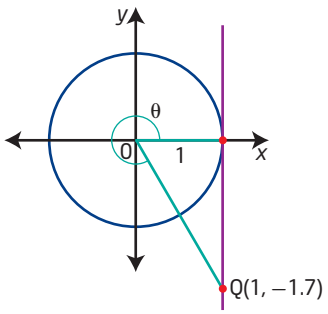

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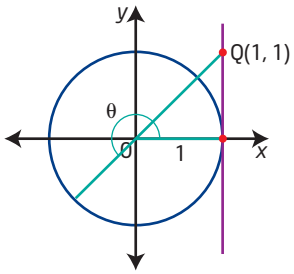
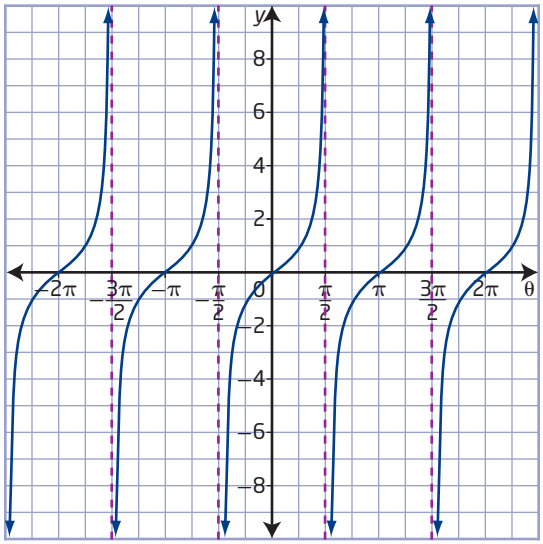


2. Use the graph of the function $y = \tan \theta$ to determine each value.



a) $\tan \frac{\pi}{2}$

b) $\tan \frac{3\pi}{4}$

c) $\tan \left(-\frac{7\pi}{4}\right)$

d) $\tan 0$

e) $\tan \pi$

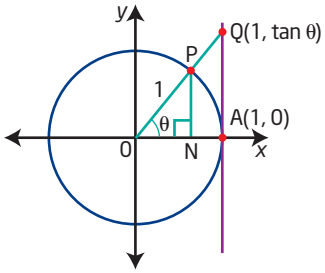
f) $\tan \frac{5\pi}{4}$

3. Does $y = \tan x$ have an amplitude? Explain.

4. Use graphing technology to graph $y = \tan x$ using the following window settings: x: $[-360°, 360°, 30°]$ and y: $[-3, 3, 1]$. Trace along the graph to locate the value of $\tan x$ when $x = 60°$. Predict the other values of x that will produce the same value for $\tan x$ within the given domain. Verify your predictions.

Apply

5. In the diagram, $\triangle PON$ and $\triangle QOA$ are similar triangles. Use the diagram to justify the statement $\tan \theta = \frac{\sin \theta}{\cos \theta}$.



6. Point $P(x, y)$ is plotted where the terminal arm of angle θ intersects the unit circle.

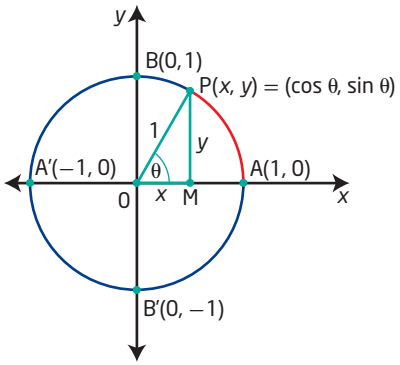
a) Use $P(x, y)$ to determine the slope of the terminal arm.

b) Explain how your result from part a) is related to $\tan \theta$.

c) Write your results for the slope from part a) in terms of sine and cosine.

d) From your answer in part c), explain how you could determine $\tan \theta$ when the coordinates of point P are known.

7. Consider the unit circle shown.



a) From $\triangle POM$, write the ratio for $\tan \theta$.

b) Use $\cos \theta$ and $\sin \theta$ to write the ratio for $\tan \theta$.

c) Explain how your answers from parts a) and b) are related.

8. The graph of $y = \tan \theta$ appears to be vertical as θ approaches 90°.

a) Copy and complete the table. Use a calculator to record the tangent values as θ approaches 90°.

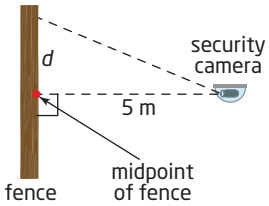
θ	$\tan \theta$
89.5°	
89.9°	
89.999°	
89.999 999°	

b) What happens to the value of $\tan \theta$ as θ approaches 90°?

c) Predict what will happen as θ approaches 90° from the other direction.

θ	$\tan \theta$
90.5°	
90.01°	
90.001°	
90.000 001°	

9. A security camera scans a long straight fence that encloses a section of a military base. The camera is mounted on a post that is located 5 m from the midpoint of the fence. The camera makes one complete rotation in 60 s.



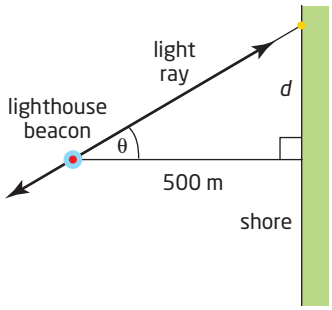
a) Determine the tangent function that represents the distance, d, in metres, along the fence from its midpoint as a function of time, t, in seconds, if the camera is aimed at the midpoint of the fence at $t = 0$.

b) Graph the function in the interval $-15 \leq t \leq 15$.

c) What is the distance from the midpoint of the fence at $t = 10$ s, to the nearest tenth of a metre?

d) Describe what happens when $t = 15$ s.

10. A rotating light on top of a lighthouse sends out rays of light in opposite directions. As the beacon rotates, the ray at angle θ makes a spot of light that moves along the shore. The lighthouse is located 500 m from the shoreline and makes one complete rotation every 2 min.



a) Determine the equation that expresses the distance, d, in metres, as a function of time, t, in minutes.

b) Graph the function in part a).

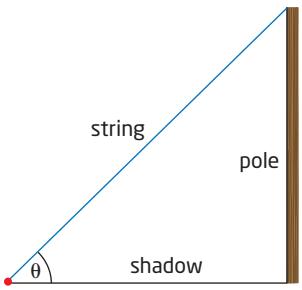
c) Explain the significance of the asymptote in the graph at $\theta = 90°$.

Did You Know?

The Fisgard Lighthouse was the first lighthouse built on Canada's west coast. It was built in 1860 before Vancouver Island became part of Canada and is located at the entrance to Esquimalt harbour.

11. A plane flying at an altitude of 10 km over level ground will pass directly over a radar station. Let d be the ground distance from the antenna to a point directly under the plane. Let x represent the angle formed from the vertical at the radar station to the plane. Write d as a function of x and graph the function over the interval $0 \leq x \leq \frac{\pi}{2}$.

12. Andrea uses a pole of known height, a piece of string, a measuring tape, and a calculator for an assignment. She places the pole in a vertical position in the school field and runs the string from the top of the pole to the tip of the shadow formed by the pole. Every 15 min, Andrea measures the length of the shadow and then calculates the slope of the string and the measure of the angle. She records the data and graphs the slope as a function of the angle.



a) What type of graph would you expect Andrea to graph to represent her data?

b) When the Sun is directly overhead and no shadow results, state the slope of the string. How does Andrea's graph represent this situation?

Extend

13. a) Graph the line $y = \frac{3}{4}x$, where $x > 0$. Mark an angle θ that represents the angle formed by the line and the positive x-axis. Plot a point with integral coordinates on the line $y = \frac{3}{4}x$.

b) Use these coordinates to determine $\tan \theta$.

c) Compare the equation of the line with your results in part b). Make a conjecture based on your findings.

14. Have you ever wondered how a calculator or computer program evaluates the sine, cosine, or tangent of a given angle? The calculator or computer program approximates these values using a power series. The terms of a power series contain ascending positive integral powers of a variable. The more terms in the series, the more accurate the approximation. With a calculator in radian mode, verify the following for small values of x, for example, $x = 0.5$.

a) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$

b) $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$

c) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$

Create Connections

C1 How does the domain of $y = \tan x$ differ from that of $y = \sin x$ and $y = \cos x$? Explain why.

C2 a) On the same set of axes, graph the functions $f(x) = \cos x$ and $g(x) = \tan x$. Describe how the two functions are related.

b) On the same set of axes, graph the functions $f(x) = \sin x$ and $g(x) = \tan x$. Describe how the two functions are related.

C3 Explain how the equation $\tan (x + \pi) = \tan x$ relates to circular functions.

5.4

Equations and Graphs of Trigonometric Functions

Focus on...

- using the graphs of trigonometric functions to solve equations
- analysing a trigonometric function to solve a problem
- determining a trigonometric function that models a problem
- using a model of a trigonometric function for a real-world situation

One of the most useful characteristics of trigonometric functions is their periodicity. For example, the times of sunsets, sunrises, and comet appearances; seasonal temperature changes; the movement of waves in the ocean; and even the quality of a musical sound can be described using trigonometric functions. Mathematicians and scientists use the periodic nature of trigonometric functions to develop mathematical models to predict many natural phenomena.

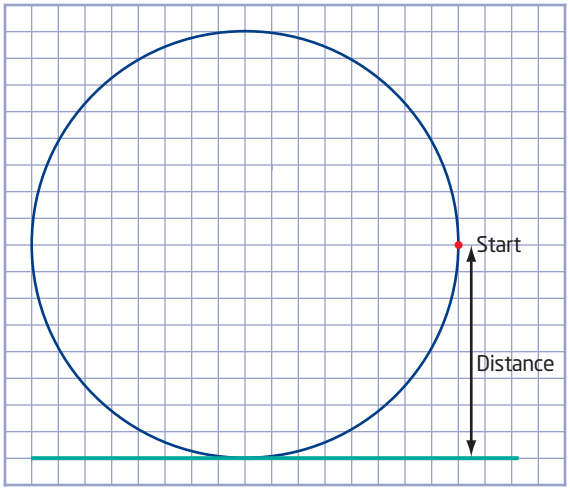
Investigate Trigonometric Equations

Materials

- marker
- ruler
- compass
- stop watch
- centimetre grid paper

Work with a partner.

1. On a sheet of centimetre grid paper, draw a circle of radius 8 cm. Draw a line tangent to the bottom of the circle.



2. Place a marker at the three o'clock position on the circle. Move the marker around the circle in a counterclockwise direction, measuring the time it takes to make one complete trip around the circle.

Aim to complete one revolution in 20 s. You may have to practice this several times to maintain a consistent speed.

3. Move the marker around the circle a second time stopping at time intervals of 2 s. Measure the vertical distance from the marker to the tangent line. Complete a table of times and distances.

Time (s)	0	2	4	6	8	10	12	14	16	18	20
Distance (cm)	8										

4. Create a scatterplot of distance versus time. Draw a smooth curve connecting the points.

Did You Know?

A scatter plot is the result of plotting data that can be represented as ordered pairs on a graph.

5. Write a function for the resulting curve.

6. **a)** From your initial starting position, move the marker around the circle in a counterclockwise direction for 3 s. Measure the vertical distance of the marker from the tangent line. Label this point on your graph.

   b) Continue to move the marker around the circle to a point that is the same distance as the distance you recorded in part a). Label this point on your graph.

   c) How do these two points relate to your function in step 5?

   d) How do the measured and calculated distances compare?

7. Repeat step 6 for other positions on the circle.

Reflect and Respond

8. What is the connection between the circular pattern followed by your marker and the graph of distance versus time?

9. Describe how the circle, the graph, and the function are related.

Link the Ideas

You can represent phenomena with periodic behaviour or wave characteristics by trigonometric functions or model them approximately with sinusoidal functions. You can identify a trend or pattern, determine an appropriate mathematical model to describe the process, and use it to make predictions (interpolate or extrapolate).

You can use graphs of trigonometric functions to solve trigonometric equations that model periodic phenomena, such as the swing of a pendulum, the motion of a piston in an engine, the motion of a Ferris wheel, variations in blood pressure, the hours of daylight throughout a year, and vibrations that create sounds.

Example 1

Solve a Trigonometric Equation in Degrees

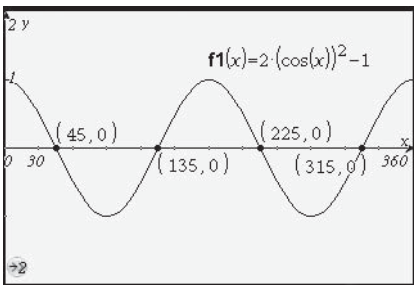
Determine the solutions for the trigonometric equation $2\cos^2 x - 1 = 0$ for the interval $0° \le x \le 360°$.

Solution

Method 1: Solve Graphically

Graph the related function $f(x) = 2\cos^2 x - 1$.

Use the graphing window [0, 360, 30] by [−2, 2, 1].



The solutions to the equation $2\cos^2 x - 1 = 0$ for the interval $0° \le x \le 360°$ are the x-intercepts of the graph of the related function.

The solutions for the interval $0° \le x \le 360°$ are $x = 45°, 135°, 225°$, and $315°$.

Method 2: Solve Algebraically

$$2\cos^2 x - 1 = 0$$
$$2\cos^2 x = 1$$
$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \pm\sqrt{\frac{1}{2}}$$

Why is the ± symbol used?

For $\cos x = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$, the angles in the interval $0° \le x \le 360°$ that satisfy the equation are $45°$ and $315°$.

For $\cos x = -\sqrt{\frac{1}{2}}$, the angles in the interval $0° \le x \le 360°$ that satisfy the equation are $135°$ and $225°$.

The solutions for the interval $0° \le x \le 360°$ are $x = 45°, 135°, 225°$, and $315°$.

Your Turn

Determine the solutions for the trigonometric equation $4\sin^2 x - 3 = 0$ for the interval $0° \le x \le 360°$.

Example 2

Solve a Trigonometric Equation in Radians

Determine the general solutions for the trigonometric equation $16 = 6 \cos \frac{\pi}{6}x + 14$. Express your answers to the nearest hundredth.

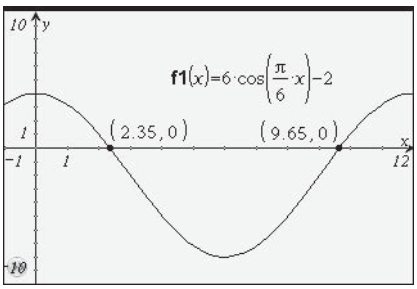
Solution

Method 1: Determine the Zeros of the Function

Rearrange the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ so that one side is equal to 0.

$6 \cos \frac{\pi}{6}x - 2 = 0$

Graph the related function $y = 6 \cos \frac{\pi}{6}x - 2$. Use the window $[-1, 12, 1]$ by $[-10, 10, 1]$.



Why should you set the calculator to radian mode?

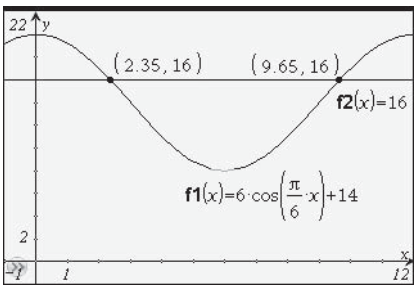
The solutions to the equation $6 \cos \frac{\pi}{6}x - 2 = 0$ are the x-intercepts.

The x-intercepts are approximately $x = 2.35$ and $x = 9.65$. The period of the function is 12 radians. So, the x-intercepts repeat in multiples of 12 radians from each of the original intercepts.

The general solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35 + 12n$ radians and $x \approx 9.65 + 12n$ radians, where n is an integer.

Method 2: Determine the Points of Intersection

Graph the functions $y = 6 \cos \frac{\pi}{6}x + 14$ and $y = 16$ using a window $[-1, 12, 1]$ by $[-2, 22, 2]$.



The solution to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ is given by the points of intersection of the curve $y = 6 \cos \frac{\pi}{6}x + 14$ and the line $y = 16$. In the interval $0 \leq x \leq 12$, the points of intersection occur at $x \approx 2.35$ and $x \approx 9.65$.

The period of the function is 12 radians. The points of intersection repeat in multiples of 12 radians from each of the original intercepts.

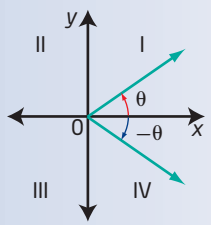
The general solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35 + 12n$ radians and $x \approx 9.65 + 12n$ radians, where n is an integer.

Method 3: Solve Algebraically

$$\begin{aligned} 16 &= 6 \cos \tfrac{\pi}{6}x + 14 \\ 2 &= 6 \cos \tfrac{\pi}{6}x \\ \tfrac{2}{6} &= \cos \tfrac{\pi}{6}x \\ \tfrac{1}{3} &= \cos \tfrac{\pi}{6}x \\ \cos^{-1}\left(\tfrac{1}{3}\right) &= \tfrac{\pi}{6}x \\ 1.2309\ldots &= \tfrac{\pi}{6}x \\ x &= 2.3509\ldots \end{aligned}$$

Did You Know?

No matter in which quadrant θ falls, $-\theta$ has the same reference angle and both θ and $-\theta$ are located on the same side of the y-axis. Since cos θ is positive on the right side of the y-axis and negative on the left side of the y-axis, $\cos \theta = \cos(-\theta)$.



Since the cosine function is positive in quadrants I and IV, a second possible value of x can be determined. In quadrant IV, the angle is $2\pi - \frac{\pi}{6}x$.

$$\begin{aligned} \tfrac{1}{3} &= \cos\left(2\pi - \tfrac{\pi}{6}x\right) \\ \cos^{-1}\left(\tfrac{1}{3}\right) &= 2\pi - \tfrac{\pi}{6}x \\ \tfrac{\pi}{6}x &= 2\pi - \cos^{-1}\left(\tfrac{1}{3}\right) \\ x &= 12 - \tfrac{6}{\pi}\cos^{-1}\left(\tfrac{1}{3}\right) \\ x &= 9.6490\ldots \end{aligned}$$

Two solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35$ and $x \approx 9.65$.

The period of the function is 12 radians, then the solutions repeat in multiples of 12 radians from each original solution.

The general solutions to the equation $16 = 6 \cos \frac{\pi}{6}x + 14$ are $x \approx 2.35 + 12n$ radians and $x \approx 9.65 + 12n$ radians, where n is an integer.

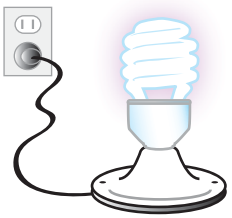
Your Turn

Determine the general solutions for the trigonometric equation $10 = 6 \sin \frac{\pi}{4}x + 8$.

Example 3

Model Electric Power

The electricity coming from power plants into your house is alternating current (AC). This means that the direction of current flowing in a circuit is constantly switching back and forth. In Canada, the current makes 60 complete cycles each second.



The voltage can be modelled as a function of time using the sine function $V = 170 \sin 120\pi t$.

a) What is the period of the current in Canada?

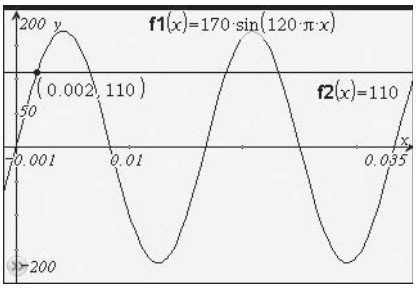
b) Graph the voltage function over two cycles. Explain what the scales on the axes represent.

c) Suppose you want to switch on a heat lamp for an outdoor patio. If the heat lamp requires 110 V to start up, determine the time required for the voltage to first reach 110 V.

Solution

a) Since there are 60 complete cycles in each second, each cycle takes $\frac{1}{60}$ s. So, the period is $\frac{1}{60}$.

b) To graph the voltage function over two cycles on a graphing calculator, use the following window settings:
x: $[-0.001, 0.035, 0.01]$
y: $[-200, 200, 50]$



The y-axis represents the number of volts. Each tick mark on the y-axis represents 50 V.

The x-axis represents the time passed. Each tick mark on the x-axis represents 0.01 s.

c) Graph the line $y = 110$ and determine the first point of intersection with the voltage function. It will take approximately 0.002 s for the voltage to first reach 110 V.

Your Turn

In some Caribbean countries, the current makes 50 complete cycles each second and the voltage is modelled by $V = 170 \sin 100\pi t$.

a) Graph the voltage function over two cycles. Explain what the scales on the axes represent.

b) What is the period of the current in these countries?

c) How many times does the voltage reach 110 V in the first second?

Did You Know?

Tidal power is a form of hydroelectric power that converts the energy of tides into electricity. Estimates of Canada's tidal energy potential off the Canadian Pacific coast are equivalent to approximately half of the country's current electricity demands.

Did You Know?

The number of cycles per second of a periodic phenomenon is called the frequency. The hertz (Hz) is the SI unit of frequency. In Canada, the frequency standard for AC is 60 Hz.

Voltages are expressed as root mean square (RMS) voltage. RMS is the square root of the mean of the squares of the values. The RMS voltage is given by $\frac{\text{peak voltage}}{\sqrt{2}}$. What is the RMS voltage for Canada?

Example 4

Model Hours of Daylight

Iqaluit is the territorial capital and the largest community of Nunavut. Iqaluit is located at latitude 63° N. The table shows the number of hours of daylight on the 21st day of each month as the day of the year on which it occurs for the capital (based on a 365-day year).

Why is the 21st day of each month chosen for the data in the table?

Hours of Daylight by Day of the Year for Iqaluit, Nunavut											
Jan 21	Feb 21	Mar 21	Apr 21	May 21	June 21	July 21	Aug 21	Sept 21	Oct 21	Nov 21	Dec 21
21	52	80	111	141	172	202	233	264	294	325	355
6.12	9.36	12.36	15.69	18.88	20.83	18.95	15.69	12.41	9.24	6.05	4.34

a) Draw a scatter plot for the number of hours of daylight, h, in Iqaluit on the day of the year, t.

b) Which sinusoidal function will best fit the data without requiring a phase shift: $h(t) = \sin t$, $h(t) = -\sin t$, $h(t) = \cos t$, or $h(t) = -\cos t$? Explain.

c) Write the sinusoidal function that models the number of hours of daylight.

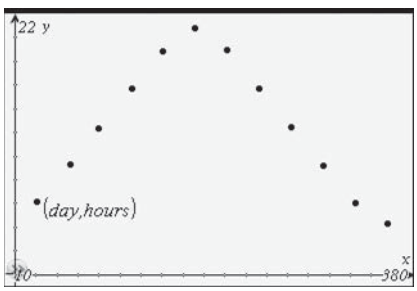
d) Graph the function from part c).

e) Estimate the number of hours of daylight on each date.

i) March 15 (day 74) **ii)** July 10 (day 191) **iii)** December 5 (day 339)

Solution

a) Graph the data as a scatter plot.



b) Note that the data starts at a minimum value, climb to a maximum value, and then decrease to the minimum value. The function $h(t) = -\cos t$ exhibits this same behaviour.

c) The maximum value is 20.83, and the minimum value is 4.34. Use these values to find the amplitude and the equation of the sinusoidal axis.

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$|a| = \frac{20.83 - 4.34}{2}$$

$$|a| = 8.245$$

The sinusoidal axis lies halfway between the maximum and minimum values. Its position will determine the value of d.

$$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$d = \frac{20.83 + 4.34}{2}$$

$$d = 12.585$$

Determine the value of b. You know that the period is 365 days.

$$\text{Period} = \frac{2\pi}{|b|}$$

Why is the period 365 days?

$$365 = \frac{2\pi}{|b|}$$

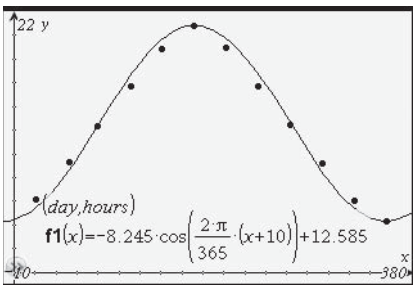
$$b = \frac{2\pi}{365}$$

Choose b to be positive.

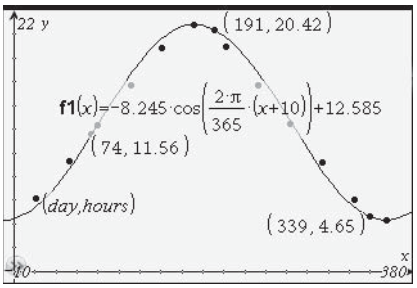
Determine the phase shift, the value of c. For $h(t) = -\cos t$ the minimum value occurs at $t = 0$. For the daylight hours curve, the actual minimum occurs at day 355, which represents a 10-day shift to the left. Therefore, $c = -10$.

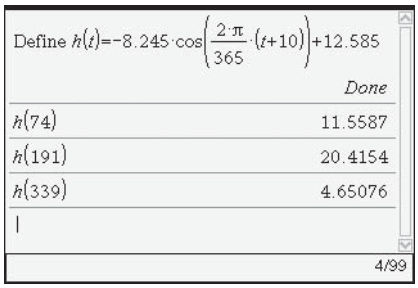
The number of hours of daylight, h, on the day of the year, t, is given by the function $h(t) = -8.245 \cos\left(\frac{2\pi}{365}(t + 10)\right) + 12.585$.

d) Graph the function in the same window as your scatter plot.



e) Use the value feature of the calculator or substitute the values into the equation of the function.





i) The number of hours of daylight on March 15 (day 74) is approximately 11.56 h.

ii) The number of hours of daylight on July 10 (day 191) is approximately 20.42 h.

iii) The number of hours of daylight on December 5 (day 339) is approximately 4.65 h.

Your Turn

Windsor, Ontario, is located at latitude 42° N. The table shows the number of hours of daylight on the 21st day of each month as the day of the year on which it occurs for this city.

Hours of Daylight by Day of the Year for Windsor, Ontario											
21	52	80	111	141	172	202	233	264	294	325	355
9.62	10.87	12.20	13.64	14.79	15.28	14.81	13.64	12.22	10.82	9.59	9.08

a) Draw a scatter plot for the number of hours of daylight, h, in Windsor, Ontario on the day of the year, t.

b) Write the sinusoidal function that models the number of hours of daylight.

c) Graph the function from part b).

d) Estimate the number of hours of daylight on each date.

i) March 10

ii) July 24

iii) December 3

e) Compare the graphs for Iqaluit and Windsor. What conclusions can you draw about the number of hours of daylight for the two locations?

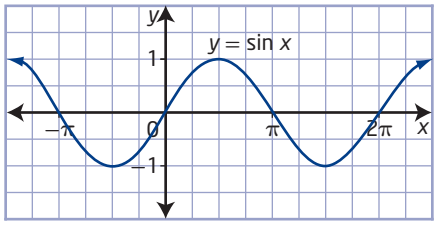
Key Ideas

- You can use sinusoidal functions to model periodic phenomena that do not involve angles as the independent variable.
- You can adjust the amplitude, phase shift, period, and vertical displacement of the basic trigonometric functions to fit the characteristics of the real-world application being modelled.
- You can use technology to create the graph modelling the application. Use this graph to interpolate or extrapolate information required to solve the problem.
- You can solve trigonometric equations graphically. Use the graph of a function to determine the x-intercepts or the points of intersection with a given line. You can express your solutions over a specified interval or as a general solution.

Check Your Understanding

Practise

1. a) Use the graph of $y = \sin x$ to determine the solutions to the equation $\sin x = 0$ for the interval $0 \le x \le 2\pi$.

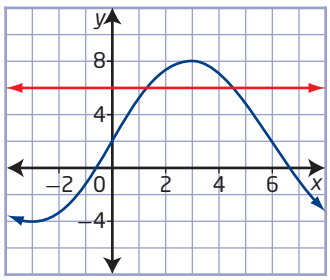


b) Determine the general solution for $\sin x = 0$.

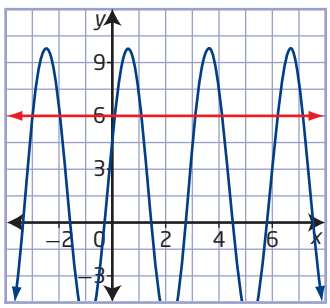
c) Determine the solutions for $\sin 3x = 0$ in the interval $0 \le x \le 2\pi$.

2. The partial sinusoidal graphs shown below are intersected by the line $y = 6$. Each point of intersection corresponds to a value of x where $y = 6$. For each graph shown determine the approximate value of x where $y = 6$.

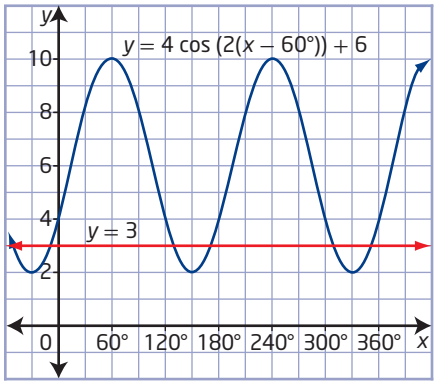
a)



b)



3. The partial graph of a sinusoidal function $y = 4 \cos (2(x - 60°)) + 6$ and the line $y = 3$ are shown below. From the graph determine the approximate solutions to the equation $4 \cos (2(x - 60°)) + 6 = 3$.



4. Solve each of the following equations graphically.

a) $-2.8 \sin \left(\frac{\pi}{6}(x - 12)\right) + 16 = 16$, $0 \le x \le 2\pi$

b) $12 \cos (2(x - 45°)) + 8 = 10$, $0° \le x \le 360°$

c) $7 \cos (3x - 18) = 4$, $0 \le x \le 2\pi$

d) $6.2 \sin (4(x + 8°)) - 1 = 4$, $0° \le x \le 360°$

5. Solve each of the following equations.

a) $\sin \left(\frac{\pi}{4}(x - 6)\right) = 0.5$, $0 \le x \le 2\pi$

b) $4 \cos (x - 45°) + 7 = 10$, $0° \le x \le 360°$

c) $8 \cos (2x - 5) = 3$, general solution in radians

d) $5.2 \sin (45(x + 8°)) - 1 = -3$, general solution in degrees

6. State a possible domain and range for the given functions, which represent real-world applications.

a) The population of a lakeside town with large numbers of seasonal residents is modelled by the function $P(t) = 6000 \sin (t - 8) + 8000$.

b) The height of the tide on a given day can be modelled using the function $h(t) = 6 \sin (t - 5) + 7$.

c) The height above the ground of a rider on a Ferris wheel can be modelled by $h(t) = 6 \sin 3(t - 30) + 12$.

d) The average daily temperature may be modelled by the function $h(t) = 9 \cos \frac{2\pi}{365}(t - 200) + 14$.

7. A trick from Victorian times was to listen to the pitch of a fly's buzz, reproduce the musical note on the piano, and say how many times the fly's wings had flapped in 1 s. If the fly's wings flap 200 times in one second, determine the period of the musical note.

8. Determine the period, the sinusoidal axis, and the amplitude for each of the following.

a) The first maximum of a sine function occurs at the point (30°, 24), and the first minimum to the right of the maximum occurs at the point (80°, 6).

b) The first maximum of a cosine function occurs at (0, 4), and the first minimum to the right of the maximum occurs at $\left(\frac{2\pi}{3}, -16\right)$.

c) An electron oscillates back and forth 50 times per second, and the maximum and minimum values occur at +10 and −10, respectively.

Apply

9. A point on an industrial flywheel experiences a motion described by the function $h(t) = 13 \cos \left(\frac{2\pi}{0.7}t\right) + 15$, where h is the height, in metres, and t is the time, in minutes.

a) What is the maximum height of the point?

b) After how many minutes is the maximum height reached?

c) What is the minimum height of the point?

d) After how many minutes is the minimum height reached?

e) For how long, within one cycle, is the point less than 6 m above the ground?

f) Determine the height of the point if the wheel is allowed to turn for 1 h 12 min.

10. Michelle is balancing the wheel on her bicycle. She has marked a point on the tire that when rotated can be modelled by the function $h(t) = 59 + 24 \sin 125t$, where h is the height, in centimetres, and t is the time, in seconds. Determine the height of the mark, to the nearest tenth of a centimetre, when $t = 17.5$ s.

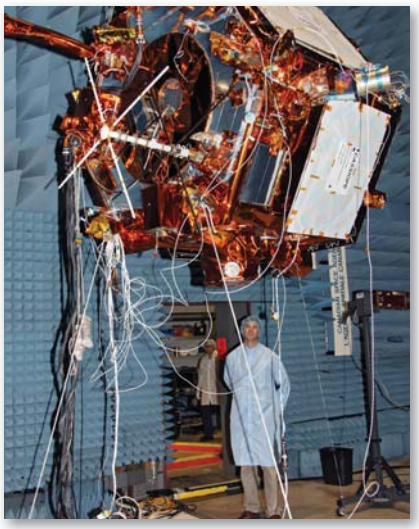
11. The typical voltage, V, in volts (V), supplied by an electrical outlet in Cuba is a sinusoidal function that oscillates between −155 V and +155 V and makes 60 complete cycles each second. Determine an equation for the voltage as a function of time, t.

12. The University of Calgary's Institute for Space Research is leading a project to launch Cassiope, a hybrid space satellite. Cassiope will follow a path that may be modelled by the function $h(t) = 350 \sin 28\pi(t - 25) + 400$, where h is the height, in kilometres, of the satellite above Earth and t is the time, in days.

a) Determine the period of the satellite.

b) How many minutes will it take the satellite to orbit Earth?

c) How many orbits per day will the satellite make?



13. The Arctic fox is common throughout the Arctic tundra. Suppose the population, F, of foxes in a region of northern Manitoba is modelled by the function $F(t) = 500 \sin \frac{\pi}{12}t + 1000$, where t is the time, in months.

a) How many months would it take for the fox population to drop to 650? Round your answer to the nearest month.

b) One of the main food sources for the Arctic fox is the lemming. Suppose the population, L, of lemmings in the region is modelled by the function $L(t) = 5000 \sin \frac{\pi}{12}(t - 12) + 10\,000$. Graph the function $L(t)$ using the same set of axes as for $F(t)$.

c) From the graph, determine the maximum and minimum numbers of foxes and lemmings and the months in which these occur.

d) Describe the relationships between the maximum, minimum, and mean points of the two curves in terms of the lifestyles of the foxes and lemmings. List possible causes for the fluctuation in populations.

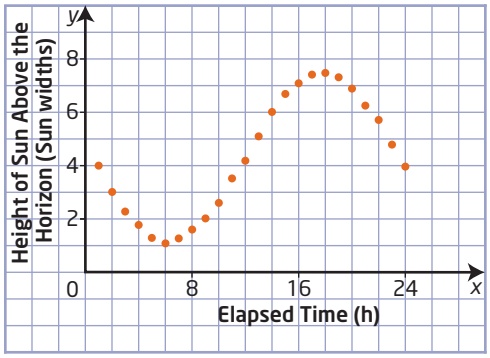
14. Office towers are designed to sway with the wind blowing from a particular direction. In one situation, the horizontal sway, h, in centimetres, from vertical can be approximated by the function $h = 40 \sin 0.526t$, where t is the time, in seconds.

a) Graph the function using graphing technology. Use the following window settings: x: [0, 12, 1], y: [−40, 40, 5].

b) If a guest arrives on the top floor at $t = 0$, how far will the guest have swayed from the vertical after 2.034 s?

c) If a guest arrives on the top floor at $t = 0$, how many seconds will have elapsed before the guest has swayed 20 cm from the vertical?

15. In Inuvik, Northwest Territories (latitude 68.3° N), the Sun does not set for 56 days during the summer. The midnight Sun sequence below illustrates the rise and fall of the polar Sun during a day in the summer.



a) Determine the maximum and minimum heights of the Sun above the horizon in terms of Sun widths.

b) What is the period?

c) Determine the sinusoidal equation that models the midnight Sun.

Did You Know?

In 2010, a study showed that the Sun's width, or diameter, is a steady 1 500 000 km. The researchers discovered over a 12-year period that the diameter changed by less than 1 km.

16. The table shows the average monthly temperature in Winnipeg, Manitoba, in degrees Celsius.

Average Monthly Temperatures for Winnipeg, Manitoba (°C)					
Jan	Feb	Mar	Apr	May	Jun
−16.5	−12.7	−5.6	3	11.3	17.3

Average Monthly Temperatures for Winnipeg, Manitoba (°C)					
Jul	Aug	Sep	Oct	Nov	Dec
19.7	18	12.5	4.5	−4.3	−11.7

a) Plot the data on a scatter plot.

b) Determine the temperature that is halfway between the maximum average monthly temperature and the minimum average monthly temperature for Winnipeg.

c) Determine a sinusoidal function to model the temperature for Winnipeg.

d) Graph your model. How well does your model fit the data?

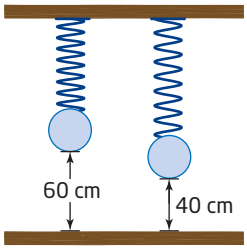
e) For how long in a 12-month period does Winnipeg have a temperature greater than or equal to 16 °C?

17. An electric heater turns on and off on a cyclic basis as it heats the water in a hot tub. The water temperature, T, in degrees Celsius, varies sinusoidally with time, t, in minutes. The heater turns on when the temperature of the water reaches 34 °C and turns off when the water temperature is 43 °C. Suppose the water temperature drops to 34 °C and the heater turns on. After another 30 min the heater turns off, and then after another 30 min the heater starts again.

a) Write the equation that expresses temperature as a function of time.

b) Determine the temperature 10 min after the heater first turns on.

18. A mass attached to the end of a long spring is bouncing up and down. As it bounces, its distance from the floor varies sinusoidally with time. When the mass is released, it takes 0.3 s to reach a high point of 60 cm above the floor. It takes 1.8 s for the mass to reach the first low point of 40 cm above the floor.



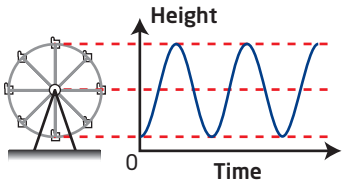
a) Sketch the graph of this sinusoidal function.

b) Determine the equation for the distance from the floor as a function of time.

c) What is the distance from the floor when the stopwatch reads 17.2 s?

d) What is the first positive value of time when the mass is 59 cm above the floor?

19. A Ferris wheel with a radius of 10 m rotates once every 60 s. Passengers get on board at a point 2 m above the ground at the bottom of the Ferris wheel. A sketch for the first 150 s is shown.



a) Write an equation to model the path of a passenger on the Ferris wheel, where the height is a function of time.

b) If Emily is at the bottom of the Ferris wheel when it begins to move, determine her height above the ground, to the nearest tenth of a metre, when the wheel has been in motion for 2.3 min.

c) Determine the amount of time that passes before a rider reaches a height of 18 m for the first time. Determine one other time the rider will be at that height within the first cycle.

20. The Canadian National Historic Windpower Centre, at Etzikom, Alberta, has various styles of windmills on display. The tip of the blade of one windmill reaches its minimum height of 8 m above the ground at a time of 2 s. Its maximum height is 22 m above the ground. The tip of the blade rotates 12 times per minute.

a) Write a sine or a cosine function to model the rotation of the tip of the blade.

b) What is the height of the tip of the blade after 4 s?

c) For how long is the tip of the blade above a height of 17 m in the first 10 s?

21. In a 366-day year, the average daily maximum temperature in Vancouver, British Columbia, follows a sinusoidal pattern with the highest value of 23.6 °C on day 208, July 26, and the lowest value of 4.2 °C on day 26, January 26.

a) Use a sine or a cosine function to model the temperatures as a function of time, in days.

b) From your model, determine the temperature for day 147, May 26.

c) How many days will have an expected maximum temperature of 21.0 °C or higher?

Extend

22. An investment company invests the money it receives from investors on a collective basis, and each investor shares in the profits and losses. One company has an annual cash flow that has fluctuated in cycles of approximately 40 years since 1920, when it was at a high point. The highs were approximately +20% of the total assets, while the lows were approximately −10% of the total assets.

a) Model this cash flow as a cosine function of the time, in years, with $t = 0$ representing 1920.

b) Graph the function from part a).

c) Determine the cash flow for the company in 2008.

d) Based on your model, do you feel that this is a company you would invest with? Explain.

23. Golden, British Columbia, is one of the many locations for heliskiing in Western Canada. When skiing the open powder, the skier leaves behind a trail, with two turns creating one cycle of the sinusoidal curve. On one section of the slope, a skier makes a total of 10 turns over a 20-s interval.

a) If the distance for a turn, to the left or to the right, from the midline is 1.2 m, determine the function that models the path of the skier.

b) How would the function change if the skier made only eight turns in the same 20-s interval?

Create Connections

C1 a) When is it best to use a sine function as a model?

b) When is it best to use a cosine function as a model?

C2 a) Which of the parameters in $y = a \sin b(x - c) + d$ has the greatest influence on the graph of the function? Explain your reasoning.

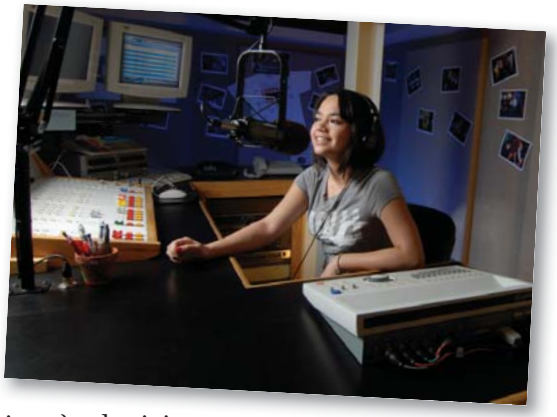
b) Which of the parameters in $y = a \cos b(x - c) + d$ has the greatest influence on the graph of the function? Explain your reasoning.

C3 The sinusoidal door by the architectural firm Matharoo Associates is in the home of a diamond merchant in Surat, India. The door measures 5.2 m high and 1.7 m wide. It is constructed from 40 sections of 254-mm-thick Burma teak. Each section is carved so that the door integrates 160 pulleys, 80 ball bearings, a wire rope, and a counterweight hidden within the single pivot. When the door is in an open position, the shape of it may be modelled by a sinusoidal function.

a) Assuming the amplitude is half the width of the door and there is one cycle created within the height of the door, determine a sinusoidal function that could model the shape of the open door.

b) Sketch the graph of your model over one period.

Project Corner Broadcasting



- Radio broadcasts, television productions, and cell phone calls are examples of electronic communication.
- A carrier waveform is used in broadcasting the music and voices we hear on the radio. The wave form, which is typically sinusoidal, carries another electrical waveform or message. In the case of AM radio, the sounds (messages) are broadcast through amplitude modulation.
- An NTSC (National Television System Committee) television transmission is comprised of video and sound signals broadcast using carrier waveforms. The video signal is amplitude modulated, while the sound signal is frequency modulated.
- Explain the difference between amplitude modulation and frequency modulation with respect to transformations of functions.
- How are periodic functions involved in satellite radio broadcasting, satellite television broadcasting, or cell phone transmissions?

Chapter 5 Review

5.1 Graphing Sine and Cosine Functions, pages 222–237

1. Sketch the graph of $y = \sin x$ for $-360° \leq x \leq 360°$.

a) What are the x-intercepts?

b) What is the y-intercept?

c) State the domain, range, and period of the function.

d) What is the greatest value of $y = \sin x$?

2. Sketch the graph of $y = \cos x$ for $-360° \leq x \leq 360°$.

a) What are the x-intercepts?

b) What is the y-intercept?

c) State the domain, range, and period of the function.

d) What is the greatest value of $y = \cos x$?

3. Match each function with its correct graph.

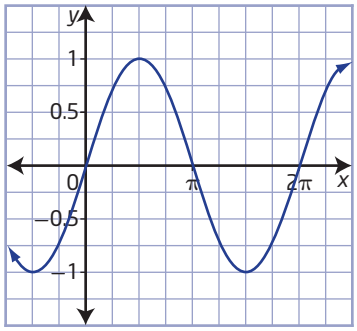
a) $y = \sin x$

b) $y = \sin 2x$

c) $y = -\sin x$

d) $y = \frac{1}{2} \sin x$

A



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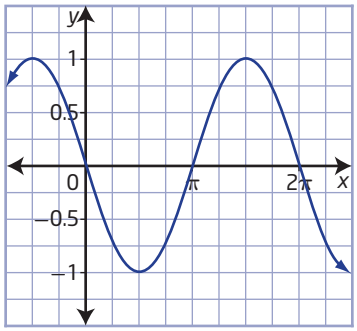

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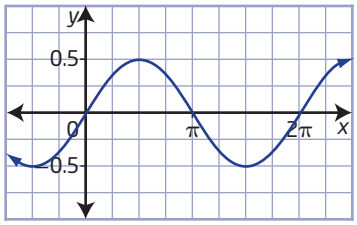

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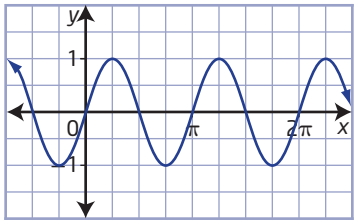


4. Without graphing, determine the amplitude and period, in radians and degrees, of each function.

a) $y = -3 \sin 2x$

b) $y = 4 \cos 0.5x$

c) $y = \frac{1}{3} \sin \frac{5}{6}x$

d) $y = -5 \cos \frac{3}{2}x$

5. a) Describe how you could distinguish between the graphs of $y = \sin x$, $y = \sin 2x$, and $y = 2 \sin x$. Graph each function to check your predictions.

b) Describe how you could distinguish between the graphs of $y = \sin x$, $y = -\sin x$, and $y = \sin (-x)$. Graph each function to check your predictions.

c) Describe how you could distinguish between the graphs of $y = \cos x$, $y = -\cos x$, and $y = \cos (-x)$. Graph each function to check your predictions.

6. Write the equation of the cosine function in the form $y = a \cos bx$ with the given characteristics.

a) amplitude 3, period π

b) amplitude 4, period 150°

c) amplitude $\frac{1}{2}$, period 720°

d) amplitude $\frac{3}{4}$, period $\frac{\pi}{6}$

7. Write the equation of the sine function in the form $y = a \sin bx$ with the given characteristics.

a) amplitude 8, period 180°

b) amplitude 0.4, period 60°

c) amplitude $\frac{3}{2}$, period 4π

d) amplitude 2, period $\frac{2\pi}{3}$

5.2 Transformations of Sinusoidal Functions, pages 238–255

8. Determine the amplitude, period, phase shift, and vertical displacement with respect to $y = \sin x$ or $y = \cos x$ for each function. Sketch the graph of each function for two cycles.

a) $y = 2 \cos 3\left(x - \frac{\pi}{2}\right) - 8$

b) $y = \sin \frac{1}{2}\left(x - \frac{\pi}{4}\right) + 3$

c) $y = -4 \cos 2(x - 30°) + 7$

d) $y = \frac{1}{3} \sin \frac{1}{4}(x - 60°) - 1$

9. Sketch graphs of the functions $f(x) = \cos 2\left(x - \frac{\pi}{2}\right)$ and $g(x) = \cos\left(2x - \frac{\pi}{2}\right)$ on the same set of axes for $0 \le x \le 2\pi$.

a) State the period of each function.

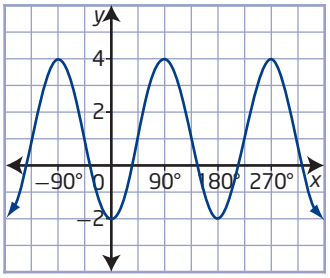
b) State the phase shift for each function.

c) State the phase shift of the function $y = \cos b(x - \pi)$.

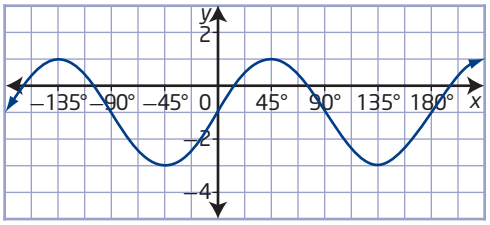
d) State the phase shift of the function $y = \cos(bx - \pi)$.

10. Write the equation for each graph in the form $y = a \sin b(x - c) + d$ and in the form $y = a \cos b(x - c) + d$.

a)



b)



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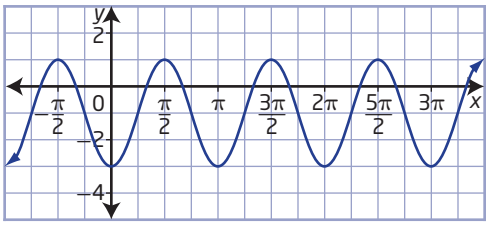
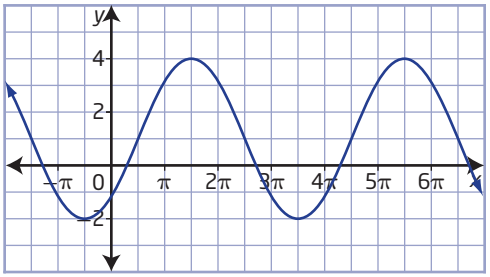


d)



11. a) Write the equation of the sine function with amplitude 4, period π, phase shift $\frac{\pi}{3}$ units to the right, and vertical displacement 5 units down.

b) Write the equation of the cosine function with amplitude 0.5, period 4π, phase shift $\frac{\pi}{6}$ units to the left, and vertical displacement 1 unit up.

c) Write the equation of the sine function with amplitude $\frac{2}{3}$, period 540°, no phase shift, and vertical displacement 5 units down.

12. Graph each function. State the domain, the range, the maximum and minimum values, and the x-intercepts and y-intercept.

a) $y = 2\cos(x - 45°) + 3$

b) $y = 4\sin 2\left(x - \frac{\pi}{3}\right) + 1$

13. Using the language of transformations, describe how to obtain the graph of each function from the graph of $y = \sin x$ or $y = \cos x$.

a) $y = 3\sin 2\left(x - \frac{\pi}{3}\right) + 6$

b) $y = -2\cos\frac{1}{2}\left(x + \frac{\pi}{4}\right) - 3$

c) $y = \frac{3}{4}\cos 2(x - 30°) + 10$

d) $y = -\sin 2(x + 45°) - 8$

14. The sound that the horn of a cruise ship makes as it approaches the dock is different from the sound it makes when it departs. The equation of the sound wave as the ship approaches is $y = 2\sin 2\theta$, while the equation of the sound wave as it departs is $y = 2\sin\frac{1}{2}\theta$.

a) Compare the two sounds by sketching the graphs of the sound waves as the ship approaches and departs for the interval $0 \le \theta \le 2\pi$.

b) How do the two graphs compare to the graph of $y = \sin\theta$?

5.3 The Tangent Function, pages 256–265

15. a) Graph $y = \tan\theta$ for $-2\pi \le \theta \le 2\pi$ and for $-360° \le \theta \le 360°$.

b) Determine the following characteristics.

i) domain

ii) range

iii) y-intercept

iv) x-intercepts

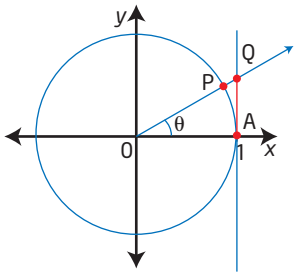
v) equations of the asymptotes

16. A point on the unit circle has coordinates $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

a) Determine the exact coordinates of point Q.

b) Describe the relationship between $\sin\theta$, $\cos\theta$, and $\tan\theta$.

c) Using the diagram, explain what happens to $\tan\theta$ as θ approaches 90°.



d) What happens to $\tan\theta$ when $\theta = 90°$?

17. a) Explain how $\cos\theta$ relates to the asymptotes of the graph of $y = \tan\theta$.

b) Explain how $\sin\theta$ relates to the x-intercepts of the graph of $y = \tan\theta$.

18. Tan θ is sometimes used to measure the lengths of shadows given the angle of elevation of the Sun and the height of a tree. Explain what happens to the shadow of the tree when the Sun is directly overhead. How does this relate to the graph of $y = \tan\theta$?

19. What is a vertical asymptote? How can you tell when a trigonometric function will have a vertical asymptote?

5.4 Equations and Graphs of Trigonometric Functions, pages 266–281

20. Solve each of the following equations graphically.

a) $2\sin x - 1 = 0,\ 0 \le x \le 2\pi$

b) $0 = 2\cos(x - 30°) + 5,\ 0° \le x \le 360°$

c) $\sin\left(\frac{\pi}{4}(x - 6)\right) = 0.5$, general solution in radians

d) $4\cos(x - 45°) + 7 = 10$, general solution in degrees

21. The Royal British Columbia Museum, home to the First Peoples Exhibit, located in Victoria, British Columbia, was founded in 1886. To preserve the many artifacts, the air-conditioning system in the building operates when the temperature in the building is greater than 22 °C. In the summer, the building's temperature varies with the time of day and is modelled by the function $T = 12 \cos t + 19$, where T represents the temperature in degrees Celsius and t represents the time, in hours.

a) Graph the function.

b) Determine, to the nearest tenth of an hour, the amount of time in one day that the air conditioning will operate.

c) Why is a model for temperature variance important in this situation?

22. The height, h, in metres, above the ground of a rider on a Ferris wheel after t seconds can be modelled by the sine function $h(t) = 12 \sin \frac{\pi}{45}(t - 30) + 15$.

a) Graph the function using graphing technology.

b) Determine the maximum and minimum heights of the rider above the ground.

c) Determine the time required for the Ferris wheel to complete one revolution.

d) Determine the height of the rider above the ground after 45 s.

23. The number of hours of daylight, L, in Lethbridge, Alberta, may be modelled by a sinusoidal function of time, t. The longest day of the year is June 21, with 15.7 h of daylight, and the shortest day is December 21, with 8.3 h of daylight.

a) Determine a sinusoidal function to model this situation.

b) How many hours of daylight are there on April 3?

24. For several hundred years, astronomers have kept track of the number of solar flares, or sunspots, that occur on the surface of the Sun. The number of sunspots counted in a given year varies periodically from a minimum of 10 per year to a maximum of 110 per year. There have been 18 complete cycles between the years 1750 and 1948. Assume that a maximum number of sunspots occurred in the year 1750.

a) How many sunspots would you expect there were in the year 2000?

b) What is the first year after 2000 in which the number of sunspots will be about 35?

c) What is the first year after 2000 in which the number of sunspots will be a maximum?

Chapter 5 Practice Test

Multiple Choice

For #1 to #7, choose the best answer.

1. The range of the function $y = 2 \sin x + 1$ is

A $\{y \mid -1 \leq y \leq 3, y \in \mathrm{R}\}$

B $\{y \mid -1 \leq y \leq 1, y \in \mathrm{R}\}$

C $\{y \mid 1 \leq y \leq 3, y \in \mathrm{R}\}$

D $\{y \mid 0 \leq y \leq 2, y \in \mathrm{R}\}$

2. What are the phase shift, period, and amplitude, respectively, for the function $f(x) = 3 \sin 2\left(x - \frac{\pi}{3}\right) + 1$?

A $\frac{\pi}{3}, 3, \pi$ **B** $\pi, \frac{\pi}{3}, 3$

C $3, \frac{\pi}{3}, \pi$ **D** $\frac{\pi}{3}, \pi, 3$

3. Two functions are given as $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ and $g(x) = \cos(x - a)$. Determine the smallest positive value for a so that the graphs are identical.

A $\frac{\pi}{4}$ **B** $\frac{\pi}{2}$ **C** $\frac{3\pi}{4}$ **D** $\frac{5\pi}{4}$

4. A cosine curve has a maximum point at (3, 14). The nearest minimum point to the right of this maximum point is (8, 2). Which of the following is a possible equation for this curve?

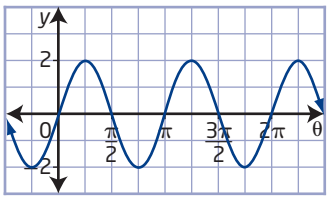
A $y = 6 \cos \frac{2\pi}{5}(x + 3) + 8$

B $y = 6 \cos \frac{2\pi}{5}(x - 3) + 8$

C $y = 6 \cos \frac{\pi}{5}(x + 3) + 8$

D $y = 6 \cos \frac{\pi}{5}(x - 3) + 8$

5. The graph of a sinusoidal function is shown. A possible equation for the function is



A $y = 2 \cos \frac{1}{2}\theta$ **B** $y = 2 \sin 2\theta$

C $y = 2 \cos 2\theta$ **D** $y = 2 \sin \frac{1}{2}\theta$

6. Monique makes the following statements about a sine function of the form $y = a \sin b(x - c) + d$:

I The values of a and d affect the range of the function.

II The values of c and d determine the horizontal and vertical translations, respectively.

III The value of b determines the number of cycles within the distance of 2π.

IV The values of a and b are vertical and horizontal stretches.

Monique's correct statements are

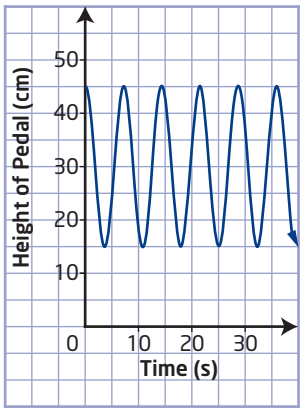
A I, II, III, and IV

B I only

C I, II, and III only

D I, II, and IV only

7. The graph shows how the height of a bicycle pedal changes as the bike is pedalled at a constant speed. How would the graph change if the bicycle were pedalled at a greater constant speed?



A The height of the function would increase.

B The height of the function would decrease.

C The period of the function would decrease.

D The period of the function would increase.

Short Answer

8. What is the horizontal distance between two consecutive zeros of the function $f(x) = \sin 2x$?

9. For the function $y = \tan \theta$, state the asymptotes, domain, range, and period.

10. What do the functions $f(x) = -4 \sin x$ and $g(x) = -4 \cos \frac{1}{2}x$ have in common?

11. An airplane's electrical generator produces a time-varying output voltage described by the equation $V(t) = 120 \sin 2513t$, where t is the time, in seconds, and V is in volts. What are the amplitude and period of this function?

12. Suppose the depth, d, in metres, of the tide in a certain harbour can be modelled by $d(t) = -3 \cos \frac{\pi}{6}t + 5$, where t is the time, in hours. Consider a day in which $t = 0$ represents the time 00:00. Determine the time for the high and low tides and the depths of each.

13. Solve each of the following equations graphically.

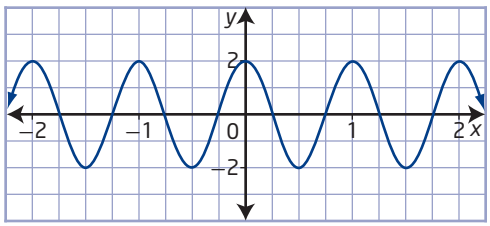
a) $\sin\left(\frac{\pi}{3}(x - 1)\right) = 0.5$, general solution in radians

b) $4 \cos (15(x + 30°)) + 1 = -2$, general solution in degrees

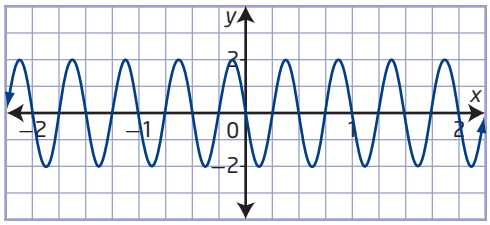
Extended Response

14. Compare and contrast the two graphs of sinusoidal functions.

I



II



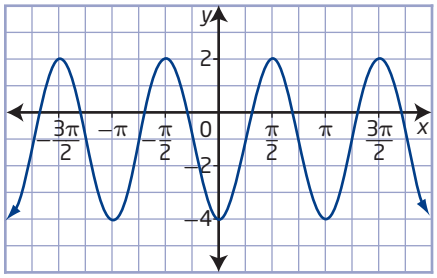
15. Suppose a mass suspended on a spring is bouncing up and down. The mass's distance from the floor when it is at rest is 1 m. The maximum displacement is 10 cm as it bounces. It takes 2 s to complete one bounce or cycle. Suppose the mass is at rest at $t = 0$ and that the spring bounces up first.

a) Write a function to model the displacement as a function of time.

b) Graph the function to determine the approximate times when the mass is 1.05 m above the floor in the first cycle.

c) Verify your solutions to part b) algebraically.

16. The graph of a sinusoidal function is shown.



a) Determine a function for the graph in the form $y = a \sin b(x - c) + d$.

b) Determine a function for the graph in the form $y = a \cos b(x - c) + d$.

17. A student is investigating the effects of changing the values of the parameters a, b, c, and d in the function $y = a \sin b(x - c) + d$. The student graphs the following functions:

A $f(x) = \sin x$

B $g(x) = 2 \sin x$

C $h(x) = \sin 2x$

D $k(x) = \sin (2x + 2)$

E $m(x) = \sin 2x + 2$

a) Which graphs have the same x-intercepts?

b) Which graphs have the same period?

c) Which graph has a different amplitude than the others?